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MATHÉMATIQUE

La solution fondamentale spéciale du système parabolique d'équations

par

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Présenté par A. MOSTOWSKI le 27 mars 1961

1. Introduction

On a défini [1] la solution spéciale fondamentale du système parabolique d'équations (voir définition (2,1), [1]). La quasi-solution spéciale (voir définition (1,1), [1]) a déjà été étudiée dans [2]. Dans le travail actuel cette solution spéciale sera étudiée en détail. Toutes les définitions des notes [1] et [2] seront maintenues, donc elles ne seront pas réproduites ici. Le système d'ordre M d'équations aux dérivées partielles sera écrit sous une forme abrégée:

$$(1,1) \qquad \hat{\Psi}^{(\alpha)}(u) = (A_\alpha(X, t) \cdot D(u))_1^P - \partial u_\alpha/\partial t \equiv 0, \qquad \alpha = 1, \dots, N, \qquad t > 0.$$

(voir (1′, 1), [1]). Ce système est défini dans un domaine Ω de l'espace euclidien de dimension n. Le cas envisagé est celui de:

$$(1', 1) \qquad n \leqslant M.$$

Les coefficients du système (1,1) sont soumis aux hypothèses suivantes:

HYPOTHÈSE I. *Les coefficients du système* (1,1) *sont continus dans l'ensemble* $(\Omega', \langle 0, T\rangle)$, *où* $T = const$ *et* Ω' *est un domaine renfermant* $\overline{\Omega}$; *ils sont aussi höldériens*:

$$(2,1) \qquad |A_{\alpha\beta}^{(k)}(X, t) - A_{\alpha\beta}^{(k)}(Y, \tau)| \leqslant \begin{cases} C_0 [|XY|^h + |t-\tau|^{h'}], & k = M, \\ C^0 |XY|^h, \quad 0 \leqslant k < M, & t = \tau, \end{cases} \quad \begin{cases} X, Y \in \Omega' \\ t, \tau \in \langle 0, T\rangle \end{cases}$$

$$k = k_1 + \dots + k_n, \quad 0 \leqslant k_i \leqslant k, \quad i = 1, \dots, n; \quad \alpha, \beta = 1, \dots, N; \quad 0 < h, h' \leqslant 1.$$

On pose: $h_1 = \min(h, Mh')$.

HYPOTHÈSE II. *Le système* (1,1) *est parabolique dans* $(\Omega', \langle 0, T\rangle)$ *selon Petrovsky* [3], *voir aussi* [4], *formule* (5).

2. L'étude des noyaux spéciaux (voir définition (1,1), [1])

En tenant compte de la formule bien utile (89), [4] de W. Pogorzelski on peut écrire:

$$(1,2)\qquad {}_AN_{\alpha\beta} = \hat{\Psi}^{(\alpha)}[{}_AW^{Y,\tau}] =$$
$$= \sum_{1\leqslant j\leqslant N}^{(M)} [A_{\alpha j}^{(M)}(X,t) - A_{\alpha j}^{(M)}(Y,\tau)]\, D^M [W_{j\beta}^{Y,\tau}(X,t;Y,\tau)] + \sum_{1\leqslant j\leqslant N}^{0\leqslant k<M} A_{\alpha j}^{(k)}(X,t)\, D^k [{}_AW_{j\beta}^{Y,\tau}(X,t;Y,\tau)] + \partial/\partial t\,({}_AP_{\alpha\beta}^{Y,\tau}), \quad \left|\begin{array}{l}\alpha,\beta = 1,\dots,N,\\ X\neq Y;\\ \tau<t;\\ |A|>0.\end{array}\right.$$

On déduit facilement de cette formule que le noyau spécial est à faible discontinuité. Il suffit, en effet, de tenir compte des hypothèses I et II ainsi que des formules (5,3) et (1,2), avec $(Z,\zeta) = (Y,\tau)$, de l'article [2] pour pouvoir écrire pour $X\in\Omega$, $Y\in\Omega'$:

$$(2,2)\qquad |{}_AN_{\alpha\beta}(X,t;Y,\tau)| \leqslant \begin{cases} \mathcal{C}^{\text{te}}(t-\tau)^{-\mu}|XY|^{-[n+M(1-\mu)-h_1]}, & 0<t-\tau<1,\ 1-h_1/M<\mu<1 \\ \mathcal{C}^{\text{te}}(t-\tau)^{-(1+1/M)}, & t-\tau\geqslant 1. \end{cases}$$

On démontrera maintenant le

LEMME 1,2. *Le noyau spécial ${}_AN_{\alpha\beta}(X,t;Y,\tau)$ est höldérien par rapport à la variable X dans chaque domaine fermé $\Omega_1\in\Omega'$ dès que la variable $Y\in\Omega'-\Omega_1$, $\tau<t<T$, les hypothèses I et II étant admises.*

Démonstration. Observons on premier lieu que le premier membre de la somme (1,2) ne diffère en rien du premier membre de la partie droite de la formule (89), [4], qui est selon M. W. Pogorzelski höldérien, les hypothèses I et II étant admises. Il suffit dès lors de démontrer que le second $i_1(X)$ et le troisième $i_2(X)$ composant de la partie droite de la formule (1,2) sont höldériens. Or, on a pour $X, X'\in\Omega_1$:

$$(3,2)\qquad i_1(X)-i_1(X') = \sum_{1\leqslant j\leqslant N}^{0\leqslant k<M} \{[A_{\alpha j}^{(k)}(X,t) - A_{\alpha j}^{(k)}(X',t)]\, D^k[{}_AW_{j\beta}^{Y,\tau}(X,t;Y,\tau)] + A_{\alpha j}^{(k)}(X',t)\, D^k[{}_AW_{j\beta}^{Y,\tau}(X,t;Y,\tau) - {}_AW_{j\beta}^{Y,\tau}(X',t;Y,\tau)]\}.$$

De là on y trouve, en appliquant l'hypothèse I et les évaluations (1,2) et (2,2), [2], que:

$$(4,2)\qquad |i_1(X)-i_1(X')| \leqslant \mathcal{C}^{\text{te}}|XX'|^h(t-\tau)^{-\mu}|XY|^{-[n+M(1-\mu)]}, \quad Y\in\Omega'-\Omega_1.$$

Les formules (5,3), [2] et (33), [4], où l'on a posé $(Z,\xi) = (Y,\tau)$, donnent l'inégalité:

$$(5,2)\qquad |i_2(X)-i_2(X')| \leqslant \mathcal{C}^{\text{te}}|XX'|(t-\tau)^{-2}\exp\{-c'[|A|/(t-\tau)^{1/M}]^q\}, \quad c'>0.$$

Comme le dernier membre de cette inégalité tend vers zéro, avec $t-\tau\to 0$, on voit que l'évaluation de l'article [4] de l'accroissement du noyau $N_{\alpha\beta}$ par rapport

à la variable spatiale X est la même que celle du noyau spécial ${}_AN_{\alpha\beta}$. Outre cela, en tenant compte des formules (1,2), [2], et (2,2), [2], on a une évaluation spéciale pour le cas où $t - \tau > 1$. De sorte qu'on peut écrire:

$$(6,2) \qquad |{}_AN_{\alpha\beta}(X, t; Y, \tau) - {}_AN_{\alpha\beta}(X', t; Y, \tau)| \leqslant$$

$$\leqslant \begin{cases} \mathcal{C}^{te}\,|XX'|^{\eta h_1}/\inf|XY|^{(n+M+1)}, & 0 < \eta < 1, \quad 0 < t - \tau < 1 \\ \mathcal{C}^{te}\,|XX'|^{h_1}/(t-\tau)^{(1+1/M)}, & t - \tau \geqslant 1 \quad X, X' \in \Omega_1, \quad Y \in \Omega' - \Omega_1 . \end{cases}$$

Le lemme (1,2) se trouve ainsi démontré.

3. Le noyau résolvant spécial

Pour définir le noyau spécial résolvant il faut définir les noyaux spéciaux, à savoir:

$$(1,3) \qquad {}_AN^{(\nu+1)}_{\alpha\beta} = {}_AN^{(\nu+1)}_{\alpha\beta}(X, t; Y, \tau) = \int_\tau^t \int_{\Omega'} S({}_AN, {}_AN^{(\nu)})\, d\Pi\, d\xi; \; {}_AN^{(0)}_{\alpha\beta} = {}_AN_{\alpha\beta},$$

$$\nu = 0, 1, \ldots .$$

Il résulte de la formule (2,2) que les noyaux (1,3) existent. De même que dans l'article [4] on démontre l'existence d'un tel indice ν_0 à partir duquel les itérations (1,3) sont bornées, les itérations à indice $\nu < \nu_0$ sont à singularités faibles. Reste l'évaluation des noyaux itérés pour l'accroissement $t - \tau > 1$. Nous allons démontrer que:

$$(2,3) \qquad |{}_AN^{(\nu)}_{\alpha\beta}| \leqslant C_\nu (t-\tau)^{-(1+1/M)}, \quad t - \tau \geqslant 1, \quad \nu = 0, 1, \ldots .$$

En effet, pour $\nu = 0$ c'est la seconde des inégalités (2,2). Observons que l'on a:

$$(3,3) \qquad \int_\tau^t \int_{\Omega'} |{}_AN_{\alpha\beta}(X, t; \Pi, \xi)|\, d\Pi\, d\xi \leqslant C, \quad t > \tau.$$

En effet, on représente l'intégrale par rapport à ξ sous la forme d'une somme de deux intégrales: avec les limites d'intégration $(\tau, t-1)$ et $(t-1, t)$. À la première intégrale la seconde des inégalités (2,2) est appliquée, à la seconde — la première. Pour démontrer l'inégalité (2,3), on écrit l'intégrale (1,3), pour $\nu = 0$, comme une somme des intégrales avec des limites d'intégration par rapport à $\xi: (\tau, (t+\tau)/2)$ et $((t+\tau)/2, t)$. On applique au premier facteur de la première intégrale la seconde des inégalités (2,2); l'intégrale restante ne surpasse pas la constante C, d'après l'inégalité (3,3). Le rôle des facteurs dans l'intégrale entre les limites $((t+\tau)/2, t)$ sera changé et l'évaluation restera la même. De sorte que l'on trouvera l'inégalité (2,3) pour $\nu = 1$. De là et de (2,2) on déduit la même inégalité que (3,3) pour le noyau à l'indice $\nu = 1$. Dès lors on peut démontrer l'inégalité (2,3) pour $\nu = 2$, et pour celle-ci l'inégalité analogue à l'inégalité (3,3), et ainsi de suite. L'inégalité (2,3) sera donc démontrée pour chaque indice $\nu = 0, 1, \ldots$.

À présent le noyau résolvant est:

$$(4,3)\qquad {}_A\mathfrak{N}_{\alpha\beta} = {}_A\mathfrak{N}_{\alpha\beta}(X, t; Y, \tau) =$$

$$= \sum_{\nu=0}^{\infty} {}_AN_{\alpha\beta}^{(\nu)}(X, t; Y, \tau), \quad \alpha, \beta = 1, \ldots, N, \ X \neq Y, \ X \in \Omega; \ Y \in \Omega'; \ 0 \leqslant \tau < t \leqslant T.$$

La convergence de la série (4,3) pour $t - \tau \leqslant 1$ ne sera démontrée que pour $X \in \Omega$, $Y \in \Omega'$ donc pour: $|XY| \leqslant \varrho = \operatorname{diam} \Omega' = \text{const}$. Comme dans ce cas l'évaluation du noyau ${}_AN_{\alpha\beta}$ est la même que celle du noyau $N_{\alpha\beta}$ dans l'article [4], la démonstration de la convergence de la série (4,3) est la même que la démonstration de la convergence de la série $\mathfrak{N}_{\alpha\beta}$ dans l'article [4] au cas spécial de $|XY| \leqslant \varrho = \text{const}$. Elle ne sera donc pas reproduite. Le cas de $t - \tau > 1$ ou de $t - \tau > T$, ne sera pas traité dans cet article. Il est utile d'observer que les singularités du noyau résolvant sont faibles et sont les mêmes que celles du noyau ${}_AN_{\alpha\beta}$.

Il reste à rappeler que la solution ${}_A\Phi_{\alpha\beta}$ du système d'équations intégrales (10,1), [1], ne diffère pas du noyau résolvant, les noyaux et les fonctions données étant identiques. Vu les formules (4,3) et (1,3) on a:

$$(5,3)\qquad {}_A\Phi_{\alpha\beta} = {}_AN_{\alpha\beta} + \int_\tau^t \int_{\Omega'} S({}_A\mathfrak{N}, {}_AN)\, d\Pi\, d\xi = {}_A\mathfrak{N}_{\alpha\beta}, \ \alpha, \beta = 1, \ldots, N.$$

On peut écrire d'après l'inégalité (2,2)

$$(6,3)\qquad |{}_A\Phi_{\alpha\beta}| = |{}_A\mathfrak{N}_{\alpha\beta}| \leqslant \mathscr{C}^{\text{te}}(t - \tau)^{-\mu} |XY|^{-[n + M(1-\mu) - h_1]},$$

$$1 - h_1/M < \mu < 1; \quad 0 < t - \tau \leqslant \text{const}; \quad X, Y \in \Omega'.$$

Nous avons le

Lemme 1,3. *Les solutions ${}_A\Phi_{\alpha\beta}$ sont höldériennes par rapport à la variable X, c'est à dire que l'on a*:

$$(7,3)\qquad |{}_A\Phi_{\alpha\beta}(X, t; Y, \tau) - {}_A\Phi_{\alpha\beta}(X', t; Y, \tau)| \leqslant$$

$$\leqslant \mathscr{C}^{\text{te}} |XX'|^{\eta h_1} \inf |XY|^{-(n+M+1)}, \quad 0 < \eta < 1; \quad X, X' \in \Omega_1; \quad Y \in \Omega' - \Omega_1.$$

En effet, on a l'égalité (10,1), [1]. Le premier composant de la partie droite de cette formule est höldérien (voir (6,2)). Il suffit de démontrer que le second composant l'est aussi. La démonstration a été faite par l'auteur de l'article [4] pour $M = 2$, $N = 1$ et $\Phi_{\alpha\beta}$ dans l'article [5]. Elle ne fut que signalée dans son article [4]. D. G. Aronson l'a reproduite complètement en 1959, (voir [6]). Il n'est pas difficile d'appliquer cette démonstration au cas du noyau ${}_AN_{\alpha\beta}$, car cette démonstration n'est basée que sur les inégalités (6,2) et (6,3) qui au cas de $t - \tau \leqslant \text{const}$ sont pour les fonctions ${}_AN_{\alpha\beta}$ et ${}_A\Phi_{\alpha\beta}$ les mêmes que pour les fonctions $N_{\alpha\beta}$ et $\Phi_{\alpha\beta}$. Nous trouvons donc démontrée l'inégalité (7,3).

4. Les propriétés de la solution fondamentale spéciale (définition (2,1), [1])

Les fonctions à intégrer dans la définition (2,1), [1], sont à singularités séparées et faibles (voir (1,2), [2] pour $Z = \Pi$, $k = 0$, et (6,3) de l'article actuel). Nous démontrerons le

Théorème (1,4). *Les hypothèses* I *et* II *étant admises, la matrice des fonctions* ${}_A\Gamma_{\alpha\beta}$ *donne les solutions du système* (1,1):

$$\hat{\Psi}^{(\alpha)}({}_A\Gamma) = 0, \quad X \neq Y, \quad X \in \Omega, \quad Y \in \Omega', \quad \alpha = 1, \ldots, N. \tag{1,4}$$

Démonstration. On a d'après la définition (2,1), [1] et (1,2):

$$\hat{\Psi}^{(\alpha)}({}_A\Gamma) = {}_AN_{\alpha\beta} + \hat{\Psi}^{(\alpha)}\left[\int_\tau^t \int_{\Omega'} S({}_AW^{\Pi,\xi}, {}_A\Phi)\, d\Pi\, d\xi\right]. \tag{2,4}$$

La fonction ${}_A\Phi_{\alpha\beta}$ étant soumise par rapport à la variable spatiale à l'inégalité de Hölder, on peut écrire le symbole $\hat{\Psi}^{(\alpha)}$ sous le signe de l'intégrale (2,4). Il faut seulement prendre en considération que la limite d'intégration est variable. On reçoit alors, en accord avec les théorèmes (3) et (4) de l'article [4], la formule suivante:

$$\hat{\Psi}^{(\alpha)}({}_A\Gamma) = {}_AN_{\alpha\beta} + \int_\tau^t \int_{\Omega'} S({}_AN, {}_A\Phi)\, d\Pi\, d\xi - \lim_{\xi\to t} \int_{\Omega'} S({}_AW^{\Pi,\xi}, {}_A\Phi)\, d\Pi. \tag{3,4}$$

Etudions la limite de l'égalité (3,4). Soit $K(X, \varrho)$ la sphère centrée au point X, de rayon ϱ suffisament petit pour que: $\varrho \leqslant |XY|/2$ et $K(X, \varrho) \in \Omega'$. Ecrivons $\Omega'' = \bar{K}(X, \varrho)\cdot\Omega'$, où $\bar{K}(X, \varrho)$ désigne le complement de $K(X, \varrho)$. Alors:

$$\int_{\Omega'} S({}_AW^{\Pi,\xi}, {}_A\Phi)\, d\Pi = \int_{K(X,\varrho)} S({}_AW^{\Pi,\xi}, {}_A\Phi)\, d\Pi + \int_{\Omega''} S({}_AW^{\Pi,\xi}, {}_A\Phi)\, d\Pi. \tag{4,4}$$

La seconde intégrale converge vers zéro, car $|X\Pi| > \varrho$, et la fonction $[{}_A\Phi]$ est intégrable. La première intégrale tend en accord avec la formule (8,3), [2], vers ${}_A\Phi_{\alpha\beta}(X, t; Y, \tau)$ car $\Pi \in K(X, \varrho)$. En effet, on a $|\Pi Y| \geqslant |XY|2$, et la fonction ${}_A\Phi_{\alpha\beta}(\Pi, \xi; Y, \tau)$ reste bornée; elle est continue pour $(\Pi, \xi) = (X, t)$. Nous avons donc:

$$\lim_{\xi\to t} \int_{\Omega'} S({}_AW^{\Pi,\xi}, {}_A\Phi)\, d\Pi = {}_A\Phi_{\alpha\beta}(X, t; Y, \tau). \tag{5,4}$$

Les formules (3,4), (5,4) et (10,1), [1] démontrent le théorème (1,4).

Il nous faut encore le

Théorème (2,4). *Les hypothèses* I *et* II *étant admises, l'intégrale généralisée de Poisson — Weierstrass existe*:

$$J_{\alpha\beta}(X, t, \tau) = \int_{\Omega'} S({}_A\Gamma, \varphi)\, dY \tag{6,4}$$

dès que la densité $[\varphi(Y, \tau)]$ *est intégrable et bornée dans* Ω'; *si elle est continue au point* (X, t), *alors*:

$$\lim_{\tau\to t} J_{\alpha\beta}(X, t, \tau) = \varphi_{\alpha\beta}(X, t). \tag{7,4}$$

Démonstration. L'existence de l'intégrale (6,4) résulte directement des formules (10,1), [1], (6,3) ainsi que de (1,2), [2], pour $k = 0$ et $Z = Y$, (voir aussi la démon-

stration de la formule (8,4). Pour démontrer la formule (7,4) observons qu'on a la définition (2,1), [1] et la formule (8,3), [2]. Il suffit donc de savoir que:

$$(8,4)\qquad \lim_{\tau\to t}\int_{\Omega'} S\left[\int_{\tau}^{t}\int_{\Omega'} S({}_AW^{\Pi,\xi},\, {}_A\Phi)\, d\Pi\, d\xi,\, \varphi\right] dY = 0 .$$

Pour démontrer la formule (8,4) il faut changer l'ordre d'intégration dans l'intégrale (8,4), prendre en considération que la fonction φ est bornée, et qu'on a les évaluations (6,3) pour $1-h_1/M<\mu'<1$ et (1,2), [2] pour $Z=\Pi$, $k=0$. On trouvera l'égalité (8,4) si l'on pose $\mu'+\mu<1$ p. exemple $\mu'=1-h_1/3M$; $\mu=h_1/3M$.

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MATHEMATICS

Some Applications of the Mixed Topology to the Theory of Two-norm Spaces

by

A. WIWEGER

Presented by E. MARCZEWSKI on March 28, 1961

The notion of a mixed topology was introduced in [8] and [9]. The mixed topology may be defined in every quasi-normal two-norm space (concerning this terminology see [3]) and, consequently, every quasi-normal two-norm space may be considered as a locally convex space. It is possible to apply in this way the theory of locally convex spaces to the theory of two-norm spaces. Some examples of such applications were given in [9]. In this paper we shall show that the theorems of Šmulian and Eberlein, well-known for Banach spaces, are also valid for γ-complete two-norm spaces. Some remarks on mixed topologies and on the extension of γ-linear functionals shall also be given.

1. Preliminaries. Extension of γ-linear functionals

Let $\langle X, \gamma\,[\tau, \tau^*]\rangle$ be a linear space with mixed topology. Throughout this paper, we shall suppose that the topology τ is finer than τ^*, that τ is defined by a homogeneous norm $\|\ \|$, and that there exists a homogeneous norm $\|\ \|_1$ equivalent to $\|\ \|$ and such that the ball $\{x : \|x\|_1 \leqslant 1\}$ is τ^*-closed (i.e. we suppose that the conditions (n), (o), (d) of [9] are satisfied). We suppose in addition that the topology τ^* is locally convex. Then the topology $\gamma\,[\tau, \tau^*]$ is also locally convex. We shall use for mixed topology the symbols $\gamma\,[\|\ \|, \tau^*]$ or τ^γ (or $\gamma\,[\|\ \|, \|\ \|^*]$, if τ^* is also defined by a norm $\|\ \|^*$). The conjugate spaces of $\langle X, \|\ \|\rangle$, $\langle X, \tau^*\rangle$ and $\langle X, \tau^\gamma\rangle$ will be denoted by Ξ, Ξ^* and Ξ^γ, respectively. If $\langle X_1, \tau_1\rangle$ is a locally convex space and if Ξ_1 is its conjugate, then the weak topology $\sigma\,(X_1, \Xi_1)$ will be denoted by $\sigma\tau_1$.

It is known that under our assumptions the space with mixed topology $\langle X, \tau^\gamma\rangle$ satisfies the conditions:

(α) The conjugate Ξ^γ with strong topology $\beta\,(\Xi^\gamma, X)$ is a Banach space.

(β) If τ_1 is a linear topology defined on X such that $\tau_1\,|Z = \tau^\gamma|\,Z$ for each τ^γ-bounded set $Z \subset X$, then τ_1 is coarser than τ^γ.

In fact, the bounded sets in spaces $\langle X, \|\ \|\rangle$ and $\langle X, \tau^\gamma\rangle$ are identical ([9], p. 56), consequently the strong topology $\beta\,(\Xi^\gamma, X)$ is identical with the topology of the norm induced on Ξ^γ by the norm defining the strong topology in Ξ ([2],

p. 280). Moreover, Ξ^γ is a closed subspace of $\langle \Xi, \| \ \| \rangle$. Hence, the condition ($\alpha$) is satisfied. Condition (β) follows by theorem 2.2 of [8].

Conversely, if $\langle X, \tau^\gamma \rangle$ is a locally convex space such that conditions (α) and (β) are satisfied, then $\langle X, \tau^\gamma \rangle$ is a space with mixed topology.

Indeed, by condition (α) the topology $\beta(\Xi^\gamma, X)$ is defined by a homogeneous norm $\| \ \|$. Let $\|x\| = \sup\{|\xi(x)| : \xi \in \Xi^\gamma, \|\xi\| \leqslant 1\}$ for each $x \in X$. If τ is the topology in X defined by the norm $\| \ \|$, then the pair of topologies τ, τ^γ satisfies the conditions (n), (o), (d) of [9] and we have $\gamma[\tau, \tau^\gamma] = \tau^\gamma$.

We see that the spaces with mixed topology can be characterized by the properties (α) and (β).

The following example shows that the property (β) does not follow from (α). Let $\langle X, \| \ \| \rangle$ be the conjugate space of Banach space $\langle Y, \| \ \| \rangle$ and let τ_1 be the weak topology $\sigma(X, Y)$. The conjugate space of $\langle X, \tau_1 \rangle$ is identical with Y and the strong topology in Y is identical with the topology defined by the norm $\| \ \|$. So the space $\langle X, \tau_1 \rangle$ satisfies condition (α). On the other hand, condition (β) is not satisfied, because $\gamma[\| \ \|, \tau_1] \neq \tau_1$, in general ([9], p. 66).

The following remark follows immediately from the theorem of Grothendieck—Alexiewicz—Semadeni ([4], ch. IV. par. 3 exerc. 3; [1], theorem 1): Let τ_1^* and τ_2^* be two locally convex topologies defined in the space $\langle X, \| \ \| \rangle$ and suppose that the pairs $(\| \ \|, \tau_1^*)$ and $(\| \ \|, \tau_2^*)$ satisfy conditions (n), (o), (d). If the conjugate spaces Ξ_1^* and Ξ_2^* of the spaces $\langle X, \tau_1^* \rangle$ and $\langle X, \tau_2^* \rangle$ are identical, then the conjugate spaces Ξ_1^γ and Ξ_2^γ of the spaces $\langle X, \gamma[\| \ \|, \tau_1^*] \rangle$ and $\langle X, \gamma[\| \ \|, \tau_2^*] \rangle$ are also identical.

1.1. *The pairs of topologies* $(\tau, \sigma\tau^*)$ *and* $(\tau, \sigma\tau^\gamma)$ *satisfy conditions* (n), (o), (d), *and*

$$\gamma[\tau, \sigma\tau^*] = \gamma[\tau, \sigma\tau^\gamma],$$

$$\sigma\gamma[\tau, \sigma\tau^*] = \sigma\gamma[\tau, \sigma\tau^\gamma] = \sigma\gamma[\tau, \tau^*].$$

First equality follows from the theorem of Grothendieck—Alexiewicz—Semadeni and from a theorem of Dixmier ([6], p. 1059). Second and third equalities follow from the first and from the preceding remark.

The space $\langle X, \gamma[\| \ \|, \| \ \|^*] \rangle$ satisfies conditions (n), (o), (d) if and only if the two-norm space $\langle X, \| \ \|, \| \ \|^* \rangle$ is quasi-normal in the sense of Alexiewicz and Semadeni [3]. If $\tau^\gamma = \gamma[\| \ \|, \| \ \|^*]$, then Ξ^γ is identical with the set of all γ-linear functionals on $\langle X, \| \ \|, \| \ \|^* \rangle$. Let X_0 be a linear subspace of a quasi normal two-norm space $\langle X, \| \ \|, \| \ \|^* \rangle$. It follows immediately from a theorem of Hahn—Banach that

1.2. *The following equality is necessary and sufficient in order that each γ-linear functional on X_0 be extensible on the whole space X with preservation of γ-linearity*:

$$\sigma\gamma[\tau | X_0, \tau^* | X_0] = \sigma(\gamma[\tau, \tau^*] | X_0)$$

(the inequality $\sigma\gamma[\tau | X_0, \tau^* | X_0] \geqslant \sigma(\gamma[\tau, \tau^*] | X_0)$ is true in every case, [9], p. 57).

The author has proved a theorem on extension of γ-linear functionals ([9], theorem 2.6.5, p. 62). Z. Semadeni generalized this theorem ([7], p. 431), using

quite different methods. Now, it will be shown that a part of theorem of Semadeni may be deduced easily by the method applied in [9]. This method requires the assumption that the whole space X is γ-reflexive (Semadeni assumed γ-reflexivity of the subspace X_0 only). More precisely, we shall prove the following theorem:

1.3. *Let $\langle X, \| \ \|, \| \ \|^* \rangle$ be a γ-reflexive two-norm space*) and let X_0 be a linear subspace of X, closed in the mixed topology $\tau^\gamma = \gamma [\| \ \|, \| \ \|^*]$. Then every γ-linear functional defined on X_0 may be extended on the whole space X with the preservation of the γ-linearity.*

Proof. It is known ([3], p. 120) that a quasi-normal space $\langle X, \| \ \|, \| \ \|^* \rangle$ is γ-reflexive if and only if the ball $S = \{x : \|x\| \leqslant 1\}$ is conditionally compact (i.e. its closure is compact) for the weak topology $\sigma\tau^\gamma$. The space X_0 is $\sigma\tau^\gamma$-closed. Taking in theorem 2.5.1 of [9] the topology $\sigma\tau^\gamma$ in place of τ^* and passing to weak topologies we get the equality

$$\sigma\gamma [\tau | X_0, \sigma\tau^\gamma | X_0] = \sigma (\gamma [\tau, \sigma\tau^\gamma] | X_0).$$

Consequently, by 1.1.

$$\begin{aligned} \sigma\gamma [\tau | X_0, \tau^* | X_0] &= \sigma\gamma [\tau | X_0, \sigma\tau^* | X_0] \\ \leqslant \sigma\gamma [\tau | X_0, \sigma\tau^\gamma | X_0] &= \sigma (\gamma [\tau, \sigma\tau^\gamma] | X_0) \\ = (\sigma\gamma [\tau, \sigma\tau^\gamma]) | X_0 &= (\sigma\gamma [\tau, \tau^*]) | X_0. \end{aligned}$$

The desired conclusion follows by the application of 1.2.

2. Theorems of Šmulian and Eberlein for two-norm spaces

The following generalizations of theorems of Šmulian and Eberlein are known ([4], ch. IV, par. 2, exerc. 13 and 15):

A) *Let $\langle X, \tau \rangle$ be a locally convex space and let Ξ be its conjugate. Suppose that the space Ξ is separable in the topology $\sigma(\Xi, X)$ (i.e. contains a denumerable dense set). Let A be a subset of the space X. Then the following statements are equivalent:*

(i) *A is sequentially compact in the topology $\sigma(X, \Xi)$, i.e. any sequence in A has a subsequence which converges to an element of X in the topology $\sigma(X, \Xi)$.*

(ii) *every countably infinite subset of A has a limit point in $\langle X, \sigma(X, \Xi) \rangle$.*

B) *Let $\langle X, \tau \rangle$ be a complete locally convex space, let Ξ be its conjugate, and let A be a subset of X. Then the condition* (ii) *is equivalent to the following one:*

(iii) *the closure of A in the space $\langle X, \sigma(X, \Xi) \rangle$ is compact.*

We shall use the following lemma:

2.1. *Let X be a linear space and suppose that two locally convex topologies τ_1 and τ_2 are defined on X. Let Ξ_1 be the conjugate space of $\langle X, \tau_1 \rangle$, and let Ξ_2 be the conjugate space of $\langle X, \tau_2 \rangle$. If $\tau_2 \geqslant \tau_1$ and if the space $\langle \Xi_1, \sigma(\Xi_1, X) \rangle$ is separable, then the space $\langle \Xi_2, \sigma(\Xi_2, X) \rangle$ is separable, too.*

In fact, $\Xi_2 \supset \Xi_1$ and $\sigma(\Xi_1, X) = \sigma(\Xi_2, X) | \Xi_1$. Since Ξ_1 is total, Ξ_1 is dense in $\langle \Xi_2, \sigma(\Xi_2, X) \rangle$, which implies the desired conclusion.

*) Concerning the definition of γ-reflexivity see [3].

2.2. (Theorem of Šmulian-Eberlein for two-norm spaces). *Let* $\langle X, \| \|, \| \|^*\rangle$ *be a quasi-normal* γ*-complete two-norm space and let* A *be a subset of* X. *Then the statements* (i), (ii) (iii) (*where* Ξ *is replaced by* Ξ^γ) *are equivalent.*

Proof. The implications (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (ii) are obvious. If the space $\langle X, \| \|, \| \|^*\rangle$ is γ-complete, then the corresponding space $\langle X, \tau^\gamma\rangle$ is complete ([9], p. 60). Thus the implication (ii) $\Rightarrow$ (iii) is an immediate consequence of theorem B).

Now we shall prove the implication (ii) $\Rightarrow$ (i). Let $\{x_n\}$ be a sequence of elements of A and let X_0 be a τ^γ-closed linear subspace of the space X spanned by the elements x_n. Let $(\Xi^\gamma)_0$ be the conjugate space of the space $\langle X_0, \tau^\gamma|X_0\rangle$. Since the space $\langle X_0, \tau^\gamma|X_0\rangle$ is separable, and the topology $\tau^*|X_0$ defined by the norm $\| \|^*$ is coarser than $\tau^\gamma|X_0$, the space $\langle X_0, \| \|^*\rangle$ is separable, too. Consequently, the space Ξ_0^* conjugate to $\langle X_0, \| \|^*\rangle$ is separable in the topology $\sigma(\Xi_0^*, X_0)$ ([5], p. 88). Therefore, by lemma 2.1., the space $\langle(\Xi^\gamma)_0, \sigma((\Xi^\gamma)_0, X_0)\rangle$ is separable. By hypothesis (ii) every infinite subset of elements x_n has a limit point $\bar{x}$ in $\langle X, \sigma(X, \Xi^\gamma)\rangle$. The space X_0 being $\sigma(X, \Xi^\gamma)$-closed, the point x belongs to X_0. Since $\sigma(X, \Xi^\gamma)|X_0 = \sigma(X_0, (\Xi^\gamma)_0)$, we infer that every infinite subset of elements x_n has a limit point in the space $\langle X_0, \sigma(X_0, (\Xi^\gamma)_0)\rangle$. By theorem A (for the space $\langle X_0, \tau^\gamma|X_0\rangle$ and the set of elements x_n) there exists a subsequence $\{x_{m_n}\}$ convergent to an element $x_0 \in X_0$ in the topology $\sigma(X_0, (\Xi^\gamma)_0)$. Therefore $x_{m_n} \to x_0$ in the topology $\sigma(X, \Xi^\gamma)$.

Remark. The γ-completeness of the space $\langle X, \| \|, \| \|^*\rangle$ was used only in the proof of implication (ii) $\Rightarrow$ (iii).

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MATHEMATICS

A Method of Steepest Ortho-descent

by

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1. Paper [1] contains an iterative method for solving linear equations in Hilbert space. The characteristic feature of this method is that the corrections improving the approximate solutions obtained at each iteration step are orthogonal to the free vector b. Thus, the speed of the convergence depends essentially on the vector b. This fact makes it possible to introduce an arbitrary vector as a parameter. This method is in general faster than the method of steepest descent investigated by L. V. Kantorovitch [2]. The present paper contains another method having the same property. The difference between these methods is the same as between the method of minimum residua investigated by M. A. Krasnoselsky and S. G. Krein [3] and the method of steepest descent mentioned above. While in the method of minimum residua we obtain the coefficients for the corrections by minimizing the norm of the residuum, to get the same in the method of steepest descent we minimize another functional as error measure. For the reason explained above we call the method presented here the method of steepest orthogonal descent or abreviated as in the haedline.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Consider the linear equation

$$Ax = b, \qquad x, b \in H. \tag{1}$$

We shall assume operator A to be selfadjoint and positive definite, or, more precisely,

$$A^* = A,$$

where A^* is the adjoint of A, and

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x),$$

where $0 < m < M < +\infty$, m and M being the minimum and maximum eigenvalues of A, respectively.

Without loss of generality one can suppose that $\|b\| = 1$.

Instead of Eq. (1) we shall consider the following equation

$$Ax = (Ax, b)\, b, \quad x \neq 0, \quad \|b\| = 1. \tag{2}$$

Thus, if x is a solution of Eq. (2), then $x/(Ax, b)$ is the solution of Eq. (1). It follows from (2) that $(Ax, b) \neq 0$, since operator A is non-singular.

Let us now define the iterative process of solving Eq. (2).

Denote by R the linear operator defined as follows:

$$z = Rx = Ax - (Ax, b)\, b, \quad x \in H, \quad \|b\| = 1. \tag{3}$$

Let x_0 be an arbitrary element of H such that

$$(x_0, b) \neq 0. \tag{4}$$

Put

$$x_{n+1} = x_n - a_n z_n, \tag{5}$$

where

$$z_n = Rx_n.$$

Thus, we have

$$z_{n+1} = z_n - a_n Rz_n. \tag{6}$$

Consider now the quadratic functional

$$F(z) = (R^{-1} z, z) \tag{7}$$

defined for all z orthogonal to b. To show that this definition makes sense we have to prove the existence of the inverse of the operator R considered only on the subspace H_b of H orthogonal to b. But this is easy to verify, since Rx is orthogonal to b for any x of H and

$$(Rz, z) = (Az, z)$$

for any z orthogonal to b. Thus, we see that operator R is positive definite and selfadjoint. Hence, it follows the existence of R^{-1} in the subspace orthogonal to b.

Denote by m_R, M_R the minimum and maximum eigenvalues of R in $H_b \perp b$.

Thus, we have

$$m_R (z, z) \leqslant (Rz, z) \leqslant M_R (z, z) \tag{8}$$

and

$$m \leqslant m_R, \quad M_R \leqslant M. \tag{9}$$

We shall now choose a_n in (5) and in (6) so as to minimize $F(z_{n+1})$, where $F(z)$ is defined by (7). Thus, we obtain

$$a_n = \frac{(z_n, z_n)}{(Rz_n, z_n)} = \frac{(z_n, z_n)}{(Az_n, z_n)},$$

and the iterative method defined in (5) and (6) becomes of the form

$$x_{n+1} = x_n - \frac{(z_n, z_n)}{(Rz_n, z_n)} z_n, \tag{10}$$

$$z_{n+1} = z_n - \frac{(z_n, z_n)}{(Rz_n, z_n)} Rz_n. \tag{11}$$

We shall show that the sequence of z_n converges to zero. We have

$$F(z_{n+1}) = F(z_n) - \frac{(z_n, z_n)^2}{(Rz_n, z_n)}.$$

Hence, we get

$$\frac{F(z_{n+1})}{F(z_n)} = 1 - \frac{(z_n, z_n)^2}{(R^{-1} z_n, z_n)(Rz_n, z_n)}.$$

Since (see [2])

$$\frac{(z_n, z_n)^2}{(R^{-1} z_n, z_n)(Rz_n, z_n)} \geqslant \frac{4M_R m_R}{(M_R + m_R)^2},$$

we obtain

(12) $$\frac{F(z_{n+1})}{F(z_n)} \leqslant \frac{(M_R - m_R)^2}{(M_R + m_R)^2}.$$

Thus, the sequence of $F(z_n)$ converges to zero at least as fast as geometric progression with the ratio

$$\frac{M_R - m_R}{M_R + m_R}.$$

Hence, it follows, by (8), that

(13) $$z_n \to 0 \quad \text{as} \quad n \to \infty.$$

Now, if for x_n defined in (10) the norm of $z_n = Rx_n$ is sufficiently small, in virtue of (13), we take

(14) $$y_n = \frac{x_n}{(Ax_n, b)} = \frac{x_n}{(x_n, Ab)}$$

as the approximate solution of Eq. (1).

It is easy to see that

(15) $$\|x_n\| \geqslant (x_0, b) \neq 0 \quad n = 0, 1, 2, \ldots .$$

In fact, we have

$$x_n = x_0 - a_0 z_0 - \ldots - a_{n-1} z_{n-1}.$$

Since

$$(z_i, b) = 0 \quad \text{for} \quad i = 0, 1, 2, \ldots,$$

we have

$$(x_n, b) = (x_0, b) \neq 0.$$

We shall now show that

(16) $$\liminf |(Ax_n, b)| = c > 0.$$

We have, by (13),

(17) $$\|z_n\|^2 = \|Ax_n\|^2 - (Ax_n, b)^2 \to 0 \quad \text{as} \quad n \to \infty.$$

If (16) were not true, we should get for a subsequence of x_n that

$$\|Ax_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, we obtain a contradiction to (15), since operator A is non-singular. It follows from (17) and (15) that

$$c \geqslant m(x_0, b), \tag{18}$$

since

$$\|Ax_n\| \geqslant m(x_0, b).$$

We shall now give an error estimate for the approximate solution. We have, by (2), (3), (14) and (18),

$$\|Ay_n - b\| = \frac{\|z_n\|}{(Ax_n, b)} \leqslant \frac{\|z_n\|}{m(x_0, b)}.$$

Hence, we get

$$\|y_n - x^*\| \leqslant \frac{\|z_n\|}{m(Ax_n, b)} \leqslant \frac{\|z_n\|}{m^2(x_0, b)},$$

where x^* ist the solution of Eq. (1), provided that

$$\|z_n\| < m(x_0, b).$$

It follows from (8) that

$$M_R^{-1}(z, z) \leqslant (R^{-1} z, z) \leqslant m_R^{-1}(z, z) \tag{19}$$

for any z orthogonal to b. Hence, relation (19) is true also for z_n, $n = 0, 1, 2, \ldots$

Thus, we get, by (19) and (12),

$$\|z_n\| \leqslant [M_R]^{1/2} \left(\frac{M_R - m_R}{M_R + m_R}\right)^n [F(z_0)]^{1/2},$$

where $F(z_0)$ is defined in (7).

Finally, we obtain

$$\|y_n - x^*\| \leqslant \frac{[M_R F(z_0)]^{1/2}}{m(Ax_n, b)} \left(\frac{M_R - m_R}{M_R + m_R}\right)^n \leqslant \frac{[M_R F(z_0)]^{1/2}}{m^2(x_0, b)} \left(\frac{M_R - m_R}{M_R + m_R}\right)^n. \tag{20}$$

It follows from (9) and (20) that the method defined in (10), (11) is in general faster than the method of steepest descent, since according to the error estimate given by Kantorovitch [2] the method of steepest descent is at least as fast as a geometric progression with the ratio

$$q = \frac{M - m}{M + m}.$$

2. As in [1], using the fact that the speed of the convergence of the process defined in (10), (11) depends on the vector b, we can introduce an arbitrary vector as a parameter.

Let y be an arbitrary fixed vector of H. If x^* is the solution of Eq. (1) then, obviously, we have

$$A(x^* + y) = Ay + b.$$

Putting

$$b' = \frac{Ay+b}{\|Ay+b\|}, \quad x = \frac{x^*+y}{\|Ay+b\|},$$

we obtain the equation

$$Ax = b'$$

instead of Eq. (1).

Then Eq. (2) can be written in the form

(21) $$Ax = (Ax, b')\, b', \quad x \neq 0, \quad \|b'\| = 1,$$

or, equivalently,

$$\|Ay+b\|^2\, Ax = (Ax, Ay+b)\,(Ay+b).$$

Hence, the solution of Eq. (1) is

$$x^* = \frac{\|Ay+b\|^2\, x}{(Ax, Ay+b)} - y,$$

where x is a non-trivial solution of Eq. (21).

Applying the same procedure as in par. 1, we obtain the approximate solutions of Eq. (21). For this purpose let us start with an initial approximate solution x_0 of Eq. (21) such that

$$(x_0, Ay+b) \neq 0.$$

This condition replaces that in (4). Define now the linear operator

$$R_y(x) = Ax - \frac{(Ax, Ay+b)}{\|Ay+b\|^2}(Ay+b).$$

The sequence of approximate solutions of Eq. (21) is defined by the following recurrent formula

$$x_{n+1} = x_n - a_n z_n,$$

where

$$z_n = R_y(x_n),$$

and

$$z_{n+1} = z_n - a_n R_y(z_n),$$

where

$$a_n = \frac{(z_n, z_n)}{(R_y(z_n), z_n)} = \frac{(z_n, z_n)}{(Az_n, z_n)}$$

since $R_y(x)$ is orthogonal to b' and

$$(R_y(z), z) = (Az, z)$$

for all elements z orthogonal to b'. We have obtained in this way a sequence of z_n convergent to the zero element of H. If the norm of z_n is sufficiently small we take

$$y_n = \frac{x_n \|Ay+b\|^2}{(Ax_n, Ay+b)} - y = \frac{x_n \|Ay+b\|^2}{(x_n, A(Ay+b))} - y$$

as the approximate solution of Eq. (1).

The error estimate is now given by the following formula which replaces (20)

$$\|y_n - x^*\| \leqslant \frac{\|Ay+b\|^2 [M_y F_y(z_0)]^{1/2}}{|(Ax_n, Ay+b)|\, m} \left(\frac{M_y - m_y}{M_y + m_y}\right)^n \leqslant$$

$$\leqslant \frac{\|Ay+b\|^2 [M_y F_y(z_0)]^{1/2}}{m_y^2 (x_0, Ay+b)} \left(\frac{M_y - m_y}{M_y + m_y}\right)^n,$$

where

$$F_y(z) = (R_y^{-1}(z), z)$$

for all elements z orthogonal to $Ay+b$; m_y and M_y are the minimum and the maximum eigenvalues of the operator $R_y(z)$ considered only on the subspace of H orthogonal to A_y+b. Then $R_y(z)$ is selfadjoint and positive definite.

Remark. Suppose that operator A is defined by the matrix

$$A = (a_{ik}), \quad i, k = 1, 2, \ldots, N,$$

Then operator R can be written in the following matrix form

$$R = (r_{ik}), \quad i, k = 1, 2, \ldots, N,$$

where

$$r_{ik} = a_{ik} - b_i c_k,$$

$$b = (b_1, b_2, \ldots, b_N); \quad c_k = (a_k, b).$$

$$a_k = (a_{1k}, a_{2k}, \ldots, a_{Nk}), \quad k = 1, 2, \ldots, N.$$

In the same way we obtain the matrix form for operator R_y:

$$R_y = (r'_{ik}), \quad i, k = 1, 2, \ldots, N,$$

where

$$r'_{ik} = a_{ik} - b'_i c'_k,$$

$$b' = \frac{Ay+b}{\|Ay+b\|} = (b'_1, b'_2, \ldots, b'_N),$$

$$c'_k = (a_k, b'), \quad k = 1, 2, \ldots, N.$$

This remark can be useful for calculations on computer.

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MATHEMATICS

A Generalization of Laguerre's Method for Functional Equations

by

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Various iterative methods of higher order have been studied for solving functional equations in Banach spaces, more precisely, for finding zero elements of non-linear functionals in such spaces. Thus, a second order iterative method of solving such equations was given in [1]. This method is actually a generalization of Newton's well known classical method, but essentially different from that given by L. V. Kantorovitch [8], [9]. A third order iterative method of solving functional equations in Banach spaces is given in [3]. This method is a generalization of the well-known method of Tchebyshev. Another class of iterative methods of higher order for finding zero elements of non-linear functionals in Banach spaces is considered in [5]. The formalism of these methods is based on the application of König's theorem. This class contains in particular Newton's method and an abstract variant of the method of tangent hyperbolas.

This paper contains a further contribution to iterative methods of higher order. Thus, an iterative method of third order is given which is a generalization of the classical method of Laguerre. Namely, Laguerre has given an elegant method which applies only to algebraic equations having all real roots. Further results concerning the application of Laguerre's method to algebraic equations have been obtained by E. Bodewig [6] and J. G. van der Corput [7]. Our abstract variant of Laguerre's method uses these results. The main argument used to examine our method is based on the application of the majorant principle. The same principle has been applied to investigate the abstract variants of the iterative methods mentioned above.

Let us consider the algebraic equation

(1) $$f(z) = 0,$$

where $f(z)$ is a real polynome of degree N having all real roots. In this case the Hessien

(2) $$H(z) = (N-1)^2 f'^2(z) - N(N-1) f(z) f''(z),$$

is non-negative. To obtain a sequence of approximate solutions of Eq. (1) we shall use the following formula of Laguerre

$$z_{k+1} = z_k - \frac{Nf(z_k)}{f'(z_k) - \sqrt{H(z_k)}} \qquad k = 0, 1, 2, \ldots. \tag{3}$$

Suppose, in addition, that the polynome $f(z)$ has all simple roots.

Let us now consider the functional equation

$$F(x) = 0, \tag{4}$$

where $f(x)$ is a non-linear functional defined on the closed sphere $S(x_0, r)$ with centre x_0 and radius r belonging to a Banach space X. Let us assume that $F(x)$ is three times continuously differentiable in the sense of Fréchet in the sphere $S(x_0, r)$ and denote by $F'(x)$, $F''(x)$ and $F'''(x)$ the first, second and third derivatives of $F(x)$, respectively.

For the sequence of linear functionals $F'(x_n)$, $x_n \in X$, we choose a sequence of elements y_n of X such that

$$\|y_n\| = 1, \qquad F'(x_n)\, y_n = \|F'(x_n)\|, \qquad n = 0, 1, 2, \ldots \tag{5}$$

provided that such a choice is possible.

Following [2], [4], [5] let us say that Eq. (4) has an algebraic majorant equation (1), if the following conditions are satisfied:

1° $|F(x_0)| \leqslant f(z_0), \qquad z_0 \geqslant 0;$

2° $\dfrac{1}{\|F'(x_0)\|} \leqslant B_0.$

where

$$B_0 = -\frac{1}{f'(z_0)} > 0;$$

3° $\|F''(x)\| \leqslant f''(z)$ if $\|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0;$

4° $\|F'''(x)\| \leqslant f'''(z)$ if $\|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0.$

We shall now construct the iterative method for solving Eq. (4) by a generalization of Laguerre's method defined by the formula (3).

The sequence of the approximate solutions of Eq. (4) is then defined as follows:

$$x_{n+1} = x_n - \frac{NF(x_n)}{\|F'(x_n)\| + \sqrt{\mathbf{H}(x_n)}}\, y_n, \qquad n = 0, 1, 2, \ldots, \tag{6}$$

where y_n are so chosen as to satisfy condition (5) and

$$\mathbf{H}(x_n) = (N-1)^2 \|F'(x_n)\|^2 - N(N-1)\, F(x_n)\, F''(x_n)\, y_n^2.$$

The following theorem gives sufficient conditions of the convergence of the process defined in (5) which are also sufficient for the existence of a solution of Eq. (4).

THEOREM. *Suppose that Eq.* (4) *has an algebraic majorant, Eq.* (1) *having all real and simple roots. If Eq.* (1) *has a root lying to the right of* z_0, *then the sequence of*

approximate solutions x_n of Eq. (4) converges to a solution x^* of this equation. The error estimate is given by the following formula:

$$\|x^* - x_n\| \leqslant z^* - z_n, \tag{7}$$

where z^* is the nearest root of Eq. (1) *lying to the right of z_0. The convergence of the process* (6) *is cubic.*

Proof. Let us observe that process (3) can be written in the following way:

$$z_{n+1} = z_n - \frac{N}{1 + \sqrt{(N-1)^2 - N(N-1)\,a_n}} \cdot \frac{f(z_n)}{f'(z_n)}, \tag{8}$$

where

$$a_n = \frac{f(z_n)\, f''(z_n)}{[f'(z_n)]^2}.$$

Analogously, we get for process (6)

$$x_{n+1} = x_n - \frac{N}{1 + \sqrt{(N-1)^2 - N(N-1)\,A_n}} \cdot \frac{F(x_n)}{\|F'(x_n)\|}\, y_n, \tag{9}$$

where

$$A_n = \frac{F(x_n)\, F''(x_n)\, y_n^2}{\|F'(x_n)\|^2}.$$

It follows from 1°—3° that

$$A_0 \leqslant a_0. \tag{10}$$

Hence, by (8) and (9), we get

$$\|x_1 - x_0\| \leqslant z_1 - z_0. \tag{11}$$

We shall now show that all conditions 1°—4° are satisfied, if element x_0 is replaced by x_1.

Using Taylor's formula in the integral form we get, by (8),

$$f(z_1) = f(z_0) - \frac{N}{1 + \sqrt{(N-1)^2 - N(N-1)\,a_0}}\, f(z_0) - \\ + \frac{1}{2}\, \frac{N^2}{[1 + \sqrt{(N-1)^2 - N(N-1)\,a_0}]^2} \cdot \frac{f^2(z_0)}{[f'(z_0)]^2}\, f''(z_0) + \\ + \frac{1}{6} \int_{z_n}^{z_1} f'''(z)\,(z_1 - z)^3\, dz.$$

Hence, we obtain, by (8), after a simple calculation

$$f(z_1) = \left[1 + (N-1)^2 - N - (N-2)\sqrt{(N-1)^2 - N(N-1)\,a_0} - N\left(\frac{1}{2}N - 1\right) a_0\right] \times \\ \times [1 + \sqrt{(N-1)^2 - N(N-1)\,a_0}\,]^{-2} f(z_0) + \frac{1}{6} \int_{z_0}^{z_1} f'''(z)\,(z_1 - z)^3\, dz.$$

Using Taylor's expansion for

$$\sqrt{1-\frac{N}{N-1}a_0},$$

we obtain

$$f(z_1)=(N-2)\left[\frac{1}{2.4}\,\frac{N^2}{(N-1)}a_0^2+\frac{1.3}{2.4.6}\,\frac{N^3}{(N-1)^2}a_0^3+\right.$$

$$\left.+\frac{1.3.5}{2.4.6.8}\,\frac{N^4}{(N-1)^3}a_0^4+\dots\right]\left[1+\sqrt{(N-1)^2-N(N-1)a_0}\right]^{-2}\times$$

$$\times f(z_0)+\frac{1}{6}\int_{z_0}^{z_1}f'''(z)(z_1-z)^3\,dz.$$

A similar representation is true for $F(x_1)$ replacing a_0 by A_0. Hence, we get, by (10),

$$\|F(x_1)\|\leqslant f(z_1).$$

Since $f''(z)\geqslant 0$, the derivative $f'(z)$ increases, still preserving the minus sign at point z_1. Further, we have

$$\|F'(x_1)\|\geqslant\|F'(x_0)\|\left(1-\frac{\|F(x_1)-F(x_0)\|}{\|F'(x_0)\|}\right).$$

Using the abstract analogue of the fundamental formula of the integral calculus, we obtain, by 3° and (11),

$$\|F'(x_1)\|\geqslant\|F'(x_0)\|\left(1-\frac{\left\|\int_{x_0}^{x_1}F''(x)\,dx\right\|}{\|F'(x_0)\|}\right)\geqslant\|F(x_0)\|\left(1-\frac{\int_{z_0}^{z_1}f''(z)\,dz}{\|F(x_0)\|}\right)\geqslant$$

$$\geqslant\|F'(x_0)\|\left(1+\frac{f'(z_1)-f'(z_0)}{f'(z_0)}\right)=\|F(x_0)\|\frac{f'(z_1)}{f'(z_0)}.$$

Hence, we obtain

$$\frac{1}{\|F'(x_1)\|}\leqslant-\frac{1}{f'(z_1)}.$$

It is easy to verify that conditions 3° and 4° are also satisfied for x_1. In fact, if

$$\|x-x_1\|\leqslant z-z_1\leqslant z'-z_1,$$

then we have, by (11),

$$\|x-x_0\|\leqslant\|x-x_1\|+\|x_1-x_0\|\leqslant z-z_1+z_1-z_0=z-z_0\leqslant z'-z_0.$$

In the same way as above we get

$$\|x_2-x_1\|\leqslant z_2-z_1.$$

Hence, we obtain by induction

$$\|x_{n+p}-x_n\|\leqslant z_{n+p}-z_n$$

for any positive integers n and p.

The sequence of z_n is increasing and converges to the nearest root lying to the right of z_0.

Hence, we conclude that there exists an element x^* of X such that the sequence of x_n converges to x^* and

$$\|x_n - x^*\| \leqslant z_n - z^*.$$

It remains to prove that x^* is a solution of Eq. (4).

In virtue of (9) we get

$$\frac{F(x_n)}{\|F'(x_n)\|} \to 0 \quad \text{as} \quad n \to \infty.$$

But the sequence of $\|F'(x_n)\|$ is bounded and it follows from the continuity of $F(x)$ that x^* is a solution of Eq. (4).

Bodewig [6] has shown that the convergence of the process in (3) is cubic if z^* is a simple root and only linear if z^* is a multiple one. Thus, the convergence of the process in (6) has at least the same character.

According to the result of van der Corput [7], if root z^* is of multiplicity q, then the process in (3) will be cubically convergent, provided that $H(z_k)$ in (3) is replaced by $\lambda H(z_k)$, where

$$\lambda = \frac{N-q}{(N-1)\,q}.$$

Hence, in this case replacing in (6) $H(x_n)$ by $\lambda H(x_n)$ we get the cubic convergence for the process in (6).

The cubic convergence of the sequence of z_n means

$$\lim_{n\to\infty} \frac{z_{n+1} - z^*}{z_n - z^*} = c \neq 0.$$

Remark. One can prove that condition (5) can be replaced by the following one

(12) $$\|y_n\| \leqslant 1 \quad \text{and} \quad |f'(z_n)| \leqslant |F'(x_n)\,y_n \leqslant \|F'(x_n)\|.$$

In this case condition 2° should be replaced by the following one

$$\frac{1}{\|F'(x_0)\|} < B_0$$

and in formula (6) we should have $F'(x_n)\,y_n$ instead of $\|F'(x_n)\|$.

Let us observe that condition (12) can be satisfied in any Banach space. On the other hand, condition (5) is not always satisfied in an arbitrary Banach space.

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THEORETICAL PHYSICS

On Algebraical Properties of Skew Tensors

by

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1. Introduction

The aim of this paper is just to collect in a compact form all information about such properties of electromagnetic field which follow from its algebraical structure. We do not pretend here to present any essentially new results; algebraical properties investigated in this paper are mostly known (e.g. [1], [2] *), [3], etc.). All what we intend to show here is a simple and compact method of the algebraical analyzis of properties of electromagnetic field.

2. Fundamental algebraical formulae

Consider a skew-symmetric real tensor:

$$f_{\alpha\beta} = -f_{\beta\alpha}, \quad f^{*}_{\beta\alpha} = f_{\alpha\beta}, \tag{2.1}$$

which may be interpreted as the electromagnetic field **).

Because of the limited number of invariants associated with this tensor it has remarkably simple algebraical properties.

Define the pure imaginary dual tensor $\check{f}^{\alpha\beta}$ as:

$$\check{f}^{\alpha\beta} = \frac{1}{2}(g)^{-\frac{1}{2}} \varepsilon^{\alpha\beta\varrho\sigma} f_{\rho\sigma} \tag{2.2}$$

(the quantity $(g)^{1/2}$ is pure imaginary!). The inverse equation reads:

$$f_{\alpha\beta} = \frac{1}{2}(g)^{-\frac{1}{2}} \varepsilon_{\alpha\beta\rho\sigma} \check{f}^{\varrho\sigma}. \tag{2.3}$$

The quantities $\check{f}$ are pseudo-tensorial quantities in the sense that their transformation law consists in the normal tensorial transformation times the sign of the Jacobian of the transformation in question; it is so because the transformed $\varepsilon^{\alpha\beta\varrho\sigma}$ goes into

*) Here see for further references.

**) The general-relativistic notation is adopted; the metric tensor $g_{\nu\mu}$ is assumed to satisfy Hilbert's condition, i.e. to correspond to the signature $(+, -, -, -)$.

itself times the Jacobian $\partial(x')/\partial(x)$, when $(g')^{1/2}=(g)^{1/2}\,|\partial(x)/\partial(x')|$. (Incidentally, when we say that $(-g)^{1/2}\,d_4x$ forms an invariant element of volume, we mean that the transformed $(-g')^{1/2}$ takes care about the absolute value of the Jacobian which one gets by changing variables of integration; that what takes care about the sign of Jacobian is the appropriate order of limits of integration.)

Now, consider two products: $\check{f}^{\alpha\varrho}f_{\rho\beta}$, $f^{\alpha\varrho}f_{\rho\beta}$. Replace in the first $\check{f}$ through f accordingly with (2.2) and f through $\check{f}$, using (2.3), respectively. In the second product let replace both f quantities through $\check{f}$ by using (2.3) (remembering, of course, that $f^{\alpha\beta}=g^{\alpha\varrho}g^{\beta\sigma}f_{\rho\sigma}$). In derived expressions enter products of two contracted ε's, i.e. $\varepsilon_{\alpha_1\alpha_2\alpha_3\rho}\,\varepsilon^{\beta_1\beta_2\beta_3\varrho}=\delta^{\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3}$. After developing the determinant for the Kronecker's δ'ta with six indices we obtain easily two identities:

$$\check{f}^{\alpha\varrho}f_{\rho\beta}=-G\,\delta^{\alpha}{}_{\beta}, \tag{2.4}$$

$$f^{\alpha\beta}f_{\rho\beta}+\check{f}^{\alpha\varrho}\check{f}_{\rho\beta}=-2F\,\delta^{\alpha}{}_{\beta}, \tag{2.5}$$

where:

$$F=\frac{1}{4}f^{\nu\mu}f_{\nu\mu},\qquad G=\frac{1}{4}\check{f}^{\nu\mu}f_{\nu\mu}. \tag{2.6}$$

(Defining $\check{F}=\frac{1}{4}\check{f}^{\nu\mu}\check{f}_{\nu\mu}$ we easily conclude from (2.5) that $\check{F}=F$; note that G is pure imaginary.) Intercombining (2.4)—(2.5) we may write our two fundamental algebraical identities in the form:

$$(f^{\alpha\varrho}\pm\check{f}^{\alpha\varrho})(f_{\rho\beta}\pm\check{f}_{\rho\beta})=-2\,\delta^{\alpha}{}_{\beta}(F\pm G). \tag{2.7}$$

One of interesting consequences of (2.6) consists in the fact that accordingly with it:

$$T^{\alpha}{}_{\beta}\underset{\mathrm{df}}{=}f^{\alpha}{}_{\rho}f^{\varrho}{}_{\beta}+F\,\delta^{\alpha}{}_{\beta}=\frac{1}{2}(f^{\alpha}{}_{\rho}f^{\varrho}{}_{\beta}-\check{f}^{\alpha}{}_{\rho}\check{f}^{\varrho}{}_{\beta}). \tag{2.8}$$

The trace-less and symmetric $T_{\alpha\beta}$ has obviously the meaning of the energy momentum tensor of the electromagnetic field $f^{\alpha\beta}$ times the factor $-4\Pi C$.

The identities (2.4)—(2.5) enable us to compute simply various "powers" of the matrix $f^{\alpha}{}_{\beta}$. Indeed:

$$\begin{aligned} f^{\alpha}{}_{\rho}f^{\varrho}{}_{\sigma}f^{\sigma}{}_{\beta} &= f^{\alpha}{}_{\rho}(-2F\delta^{\varrho}{}_{\beta}-\check{f}^{\varrho}{}_{\sigma}\check{f}^{\sigma}{}_{\beta}) \\ &= -2Ff^{\alpha}{}_{\beta}+(-f^{\alpha}{}_{\rho}\check{f}^{\varrho}{}_{\sigma})\check{f}^{\sigma}{}_{\beta} \\ &= -2Ff^{\alpha}{}_{\beta}+G\check{f}^{\alpha}{}_{\beta}. \end{aligned} \tag{2.9}$$

Therefore, the third power of f is linearly expressible by f and $\check{f}$.

There holds, for the same reasons:

$$\check{f}^{\alpha}{}_{\rho}\check{f}^{\varrho}{}_{\sigma}\check{f}^{\sigma}{}_{\beta}=-2F\check{f}^{\alpha}{}_{\beta}+Gf^{\alpha}{}_{\beta}. \tag{2.10}$$

Multiplying (2.9)—(2.10) by $f^{\gamma}{}_{\alpha}$, $\check{f}^{\gamma}{}_{\alpha}$ respectively and applying again (2.4) one gets:

$$f^{\alpha}{}_{\rho}\, f^{\varrho}{}_{\sigma}\, f^{\sigma}{}_{\tau}\, f^{\tau}{}_{\beta} + 2F f^{\alpha}{}_{\rho}\, f^{\varrho}{}_{\beta} + G^2\, \delta^{\alpha}{}_{\beta} = 0, \tag{2.11}$$

$$\check{f}^{\alpha}{}_{\rho}\, \check{f}^{\varrho}{}_{\sigma}\, \check{f}^{\sigma}{}_{\tau}\, \check{f}^{\tau}{}_{\beta} + 2F \check{f}^{\alpha}{}_{\rho}\, \check{f}^{\varrho}{}_{\beta} + G^2\, \delta^{\alpha}{}_{\beta} = 0. \tag{2.12}$$

Now, it is obvious to see that in terms of $T^{\alpha}{}_{\beta}$ defined by (2.8) the Eq. (2.11) may be rewritten as:

$$T^{\alpha}{}_{\rho}\, T^{\varrho}{}_{\beta} = \delta^{\alpha}{}_{\beta}\,(F^2 - G^2) \tag{2.13}$$

which expresses the well known fact that the square of the energy-momentum tensor of the electromagnetic field forms a diagonal tensor.

Now, suppose that $W^{\alpha}_{(i)}$ is an eigenvector of $f^{\alpha}{}_{\beta}$ associated to the eigenvalue λ_i $(i = 1, 2, 3, 4)$. One can see at once from (2.11) that eigenvalues λ are roots of the simple secular equation:

$$\lambda^4 + 2F\lambda^2 + G^2 = 0\,. \tag{2.14}$$

Define invariants R and Φ through equation:

$$F + G = \frac{1}{2} R^2\, e^{2i\Phi}, \tag{2.15}$$

(where $R > 0$), obviously, G is a pseudo-quantity.

In terms of R and Φ, F and G may be written as:

$$F = \frac{1}{2} R^2 \cos 2\Phi, \qquad G = \frac{i}{2} R^2 \sin 2\Phi,$$

and appropriately ordered roots of (2.14) are simply:

$$\lambda_1 = iR\cos\Phi, \qquad \lambda_2 = -\,iR\cos\Phi, \qquad \lambda_3 = R\sin\Phi, \qquad \lambda^4 = -\,R\sin\Phi. \tag{2.16}$$

It is interesting to observe that with the help of (2.4) one can prove at once that the previously defined eigenvectors of $f^{\alpha}{}_{\beta}$, W^{α}_{i} are simultaneously eigenvectors of $\check{f}^{\alpha}{}_{\beta}$ corresponding, respectively, to eigenvalues:

$$\check{\lambda}_1 = \lambda_4, \qquad \check{\lambda}_2 = \lambda_3, \qquad \check{\lambda}_3 = \lambda_2, \qquad \check{\lambda}_4 = \lambda_1\,. \tag{2.17}$$

3. Projection operators

Eqs. (2.11)—(2.12) may be factorized as follows:

$$(f^{\alpha}{}_{\rho} - \lambda_{i_1}\delta^{\alpha}{}_{\rho})\,(f^{\varrho}{}_{\sigma} - \lambda_{i_2}\delta^{\varrho}{}_{\sigma})\,(f^{\sigma}{}_{\tau} - \lambda_{i_3}\delta^{\sigma}{}_{\tau})\,(f^{\tau}{}_{\beta} - \lambda_{i_4}\delta^{\tau}{}_{\beta}) = 0, \tag{3.1}$$

$$(\check{f}^{\alpha}{}_{\rho} - \check{\lambda}_{i_1}\delta^{\alpha}{}_{\rho})\,(\check{f}^{\varrho}{}_{\sigma} - \check{\lambda}_{i_2}\delta^{\varrho}{}_{\sigma})\,(\check{f}^{\sigma}{}_{\tau} - \check{\lambda}_{i_3}\delta^{\sigma}{}_{\tau})\,(\check{f}^{\tau}{}_{\beta} - \check{\lambda}_{i_4}\delta^{\tau}{}_{\beta}) = 0, \tag{3.2}$$

where i_1, i_2, i_3, i_4 are any permutation of 1, 2, 3, 4; it is true because coefficients at various "powers" of the matrix $f^{\alpha}{}_{\beta}$ in the left-hand members of (3.1), (3.2) are symmetric functions of $\lambda_{i_1}, \ldots, \lambda_{i_4}$; hence, the order of factors in (3.1), (3.2) is arbitrary.

This suggests the idea of introducing the projection operators:

$$(3.3) \qquad P_i{}^\alpha{}_\beta \underset{\mathrm{df}}{=} \frac{(f^\alpha{}_\rho - \lambda_{i_1}\delta^\alpha{}_\rho)(f^\varrho{}_\sigma - \lambda_{i_2}\delta^\varrho{}_\sigma)(f^\sigma{}_\beta - \lambda_{i_3}\delta^\sigma{}_\beta)}{(\lambda_i - \lambda_{i_1})(\lambda_i - \lambda_{i_2})(\lambda_i - \lambda_{i_3})}.$$

where i_1, i_2, i_3 are integers which together with i form any permutation of 1, 2, 3, 4. Because of the symmetry of the right hand member of (3.3) the values of P's do not depend on any particular choice of i's.

One may also consider quantities $\check{P}_i{}^\alpha{}_\beta$ constructed in the precisely the same way out of $\check{f}$'s and $\check{\lambda}$'s.

It is easily to see that Pi's are indeed projection operators. As a matter of fact it is clear from (3.1) that:

$$(3.4) \qquad P_i{}^\alpha{}_\rho\, P_j{}^\varrho{}_\beta = 0 \quad \text{if} \quad i \neq j.$$

On the other hand, because again accordingly with (3.1) the quantity $P_i{}^\alpha{}_\beta$ multiplied by $f^\varrho{}_\alpha - \lambda_i\,\delta^\varrho{}_\alpha$ or $f^\beta{}_\rho - \lambda_i\,\delta^\beta{}_\rho$ vanishes, it follows that:

$$(3.5) \qquad f^\alpha{}_\rho\, P_i{}^\varrho{}_\beta = P_i{}^\alpha{}_\rho\, f^\varrho{}_\beta = \lambda_i\, P_i{}^\alpha{}_\beta.$$

Using this result to the successive factors of $P_i{}^\varrho{}_\beta$ in $P_i{}^\alpha{}_\rho\, P_i{}^\varrho{}_\beta$ one gets at once that:

$$(3.6) \qquad P_i{}^\alpha{}_\rho\, P_i{}^\varrho{}_\beta = P_i{}^\alpha{}_\beta.$$

Of course, precisely parallel to (3.4, the formulae (3.5) and (3.6) hold for $\check{P}_i{}^\alpha{}_\beta$ because of (3.2).

Our projection operators may be represented in a more explicit form. Namely, performing the multiplication on the right hand side of (3.3), applying (2.10) with respect to the third power of f, and, subsequently, substituting the explicit values of λ_i accordingly with (2.16) we obtain:

$$(3.7) \qquad \begin{cases} P_1{}^\alpha{}_\beta = \dfrac{1}{4}\left(\delta^\alpha{}_\beta - \dfrac{2}{R^2}T^\alpha{}_\beta\right) + \dfrac{1}{2\,iR}(\cos\Phi\, f^\alpha{}_\beta - i\sin\Phi\,\check{f}^\alpha{}_\beta), \\[2ex] P_2{}^\alpha{}_\beta = \dfrac{1}{4}\left(\delta^\alpha{}_\beta - \dfrac{2}{R^2}T^\alpha{}_\beta\right) - \dfrac{1}{2\,iR}(\cos\Phi\, f^\alpha{}_\beta - i\sin\Phi\,\check{f}^\alpha{}_\beta), \\[2ex] P_3{}^\alpha{}_\beta = \dfrac{1}{4}\left(\delta^\alpha{}_\beta + \dfrac{2}{R^2}T^\alpha{}_\beta\right) + \dfrac{1}{2\,R}(\sin\Phi\, f^\alpha{}_\beta + i\cos\Phi\,\check{f}^\alpha{}_\beta), \\[2ex] P_4{}^\alpha{}_\beta = \dfrac{1}{4}\left(\delta^\alpha{}_\beta + \dfrac{2}{R^2}T^\alpha{}_\beta\right) - \dfrac{1}{2\,R}(\sin\Phi\, f^\alpha{}_\beta + i\cos\Phi\,\check{f}^\alpha{}_\beta). \end{cases}$$

Studying these formulae one can see at once that our projection operators have some interesting properties. First there holds

$$(3.8) \qquad P_i{}^\alpha{}_\alpha = 1.$$

Secondly
$$\sum_{i=1}^{4} P_i{}^\alpha{}_\beta = \delta^\alpha{}_\beta .$$

Thirdly, there hold the following decomposition formulae:

$$(3.9)\quad \begin{cases} f^\alpha{}_\beta = (iR\cos\Phi)\,P_1{}^\alpha{}_\beta + (-iR\cos\Phi)\,P_2{}^\alpha{}_\beta + (R\sin\Phi)\,P_3{}^\alpha{}_\beta + (-R\sin\Phi)\,P_4{}^\alpha{}_\beta \\ \check{f}^\alpha{}_\beta = (-R\sin\Phi)\,P_1{}^\alpha{}_\beta + (R\sin\Phi)\,P_2{}^\alpha{}_\beta + (-iR\cos\Phi)\,P_3{}^\alpha{}_\beta + (iR\cos\Phi)\,P_4{}^\alpha{}_\beta, \\ T^\alpha{}_\beta = -\frac{R^2}{2}P_1{}^\alpha{}_\beta - \frac{R^2}{2}P_2{}^\alpha{}_\beta + \frac{R^2}{2}P_3{}^\alpha{}_\beta + \frac{R^2}{2}P_4{}^\alpha{}_\beta . \end{cases}$$

It follows directly from the second formula (3.9) that:

$$(3.10)\qquad \check{P}_i{}^\alpha{}_\beta = P_i{}^\alpha{}_\beta .$$

Note that the last equation holds as the result of the order of eigenvalues $\check{\lambda}_i$ which we assumed accordingly with (2.17).

Now, raising and lowering, respectively, indices and using the fact that $T_{\alpha\beta}$ is symmetric, $f_{\alpha\beta}$, $\check{f}_{\alpha\beta}$ are skew, one gets:

$$(3.11)\qquad P_1{}_\alpha{}^{\cdot\beta}_{\cdot} = P_2{}^\beta{}_\alpha, \quad P_2{}_\alpha{}^{\cdot\beta}_{\cdot} = P_1{}^\beta{}_\alpha, \quad P_3{}_\alpha{}^{\cdot\beta}_{\cdot} = P_4{}^\beta{}_\alpha, \quad P_4{}_\alpha{}^{\cdot\beta}_{\cdot} = P_3{}^\beta{}_\alpha .$$

Moreover, our operators $P_i{}^\alpha{}_\beta$ are complex; complex conjugated to $P_i{}^\alpha{}_\beta$ are respectively:

$$(3.12)\quad (P_1{}^\alpha{}_\beta)^* = P_2{}^\alpha{}_\beta, \quad (P_2{}^\alpha{}_\beta)^* = P_1{}^\alpha{}_\beta, \quad (P_3{}^\alpha{}_\beta)^* = P_3{}^\alpha{}_\beta, \quad (P_4{}^\alpha{}_\beta)^* = P_4{}^\alpha{}_\beta .$$

As far as the tensorial properties of P's are concerned it is easy to see that $P_1{}^\alpha{}_\beta$ and $P_2{}^\alpha{}_\beta$ are tensors; the pseudo quantities $\check{f}^\alpha{}_\beta$ enter there multiplied by the pseudoscalar $\sin\Phi$ (in agreement with (2.15) R is a scalar, Φ is a pseudoscalar).

But $P_3{}^\alpha{}_\beta$, and $P_4{}^\alpha{}_\beta$ are mixtures of tensors and pseudo-tensors; under the transformation with the negative Jacobian they transform as

$$(3.13)\qquad P_3{}^\alpha{}_\beta \to P_4{}^\alpha{}_\beta; \quad P_4{}^\alpha{}_\beta \to P_3{}^\alpha{}_\beta .$$

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THEORETICAL PHYSICS

On Some Applications of Algebraical Properties of Skew Tensors

by

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Presented by L. INFELD on May 27, 1961

This paper forms the second part of [1]. The knowledge of that paper is assumed. I and II refer to formulae in [1] and this paper, respectively.

1. A simple application

Our projection operators form a very convenient mathematical tool in the elementary problem of the motion of an electron in the constant electromagnetic field.

Understanding by dots derivatives with respect to the proper time, Lorenz equations may be written as

$$\ddot{a}^\alpha = \frac{e}{mc^2} f^\alpha{}_\beta \dot{a}^\beta, \quad \dot{a}^\alpha \dot{a}_\alpha = 1. \tag{II.1.1}$$

Suppose that $f^\alpha{}_\beta$ is constant; co-ordinates used are cartesian co-ordinates in the flat Minkowski space-time.

It is evident from (II.1.1.) that its general solution which satisfies initial conditions $\dot{a}^\alpha(0) = \dot{a}^\alpha{}_0$ is:

$$\dot{a}^\alpha(s) = \left[\exp\left(\frac{es}{mc^2} f\cdot\right)\right]^\alpha{}_\beta \dot{a}^\beta{}_0, \tag{II.1.2}$$

where the exponential function of a matrix is understood as the appropriate power series of the given matrix.

Now, we must distinguish two cases: if the field R is a null field ($F = 0$, $G = 0$), accordingly with (I.2.9) and (I.2.1) all powers of $f^\alpha{}_\beta$ higher than the second vanish identically. Therefore the solution (II.1.2.) is given as

$$\dot{a}^\alpha(s) = \left[\delta^\alpha_\beta + \frac{es}{mc^2} f^\alpha{}_\beta + \frac{e^2 s^2}{2m^2 c^4} f^\alpha{}_\rho f^\rho{}_\beta\right] \dot{a}^\beta{}_0. \tag{II.1.3}$$

But in the case of a not-null field ($R \neq 0$) we may use sensibly our projection operators. Representing in (II.1.2.) the matrix $f^\alpha{}_\beta$ accordingly with the first of

(I.3.9) and using the properties of projectors (I.3.4) and (I.3.6) one obtains at once

$$\text{(II.1.4.)}\quad \dot{a}^{\alpha}(s)=[e^{\frac{ies}{mc^2}R\cos\Phi}P_1{}^{\alpha}{}_{\beta}+e^{-\frac{ies}{mc^2}R\cos\Phi}P_2{}^{\alpha}{}_{\beta}$$
$$e^{\frac{es}{mc^2}R\sin\Phi}P_3{}^{\alpha}{}_{\beta}+e^{-\frac{es}{mc^2}R\sin\Phi}P_4{}^{\alpha}{}_{\beta}]\,\dot{a}_0{}^{\beta}.$$

It may be also rewritten, by taking (I.3.7), as

$$\text{(II.1.5.)}\quad \dot{a}^{\alpha}(s)=\left\{\frac{1}{2}\left(\delta^{\alpha}{}_{\beta}-\frac{2}{R^2}T^{\alpha}{}_{\beta}\right)\cos\left[\frac{es}{mc^2}R\cos\Phi\right]\right.$$
$$\frac{1}{2}\left(\delta^{\alpha}{}_{\beta}+\frac{2}{R^2}T^{\alpha}{}_{\beta}\right)\mathrm{ch}\left[\frac{es}{mc^2}R\sin\Phi\right]$$
$$\frac{1}{R}(\cos\Phi\, f^{\alpha}{}_{\beta}-i\sin\Phi\,\check{f}^{\alpha}{}_{\beta})\sin\left[\frac{es}{mc^2}R\cos\Phi\right]$$
$$\left.\frac{1}{R}(\sin\Phi\, f^{\alpha}{}_{\beta}+i\cos\Phi\,\check{f}^{\alpha}{}_{\beta})\,\mathrm{sh}\left[\frac{es}{mc^2}R\sin\Phi\right]\right\}\dot{a}_0{}^{\beta}.$$

which forms the explicit solution in a covariant form.

The formulae (II.1.4.) may be also compactly written as

$$\text{(II.1.6.)}\qquad \dot{a}^{\alpha}(s)=\sum_{j=1}^{4}e^{\frac{ies}{mc^2}\lambda j}P_j{}^{\alpha}{}_{\beta}\,\dot{a}^{\beta}{}_0.$$

For vectors $\dot{a}_i{}^{0\alpha}=P_{i\alpha\beta}\,\dot{a}^{\beta}{}_0$, have interesting properties. One can easily check that

$$\text{(II.1.7)}\quad \dot{a}^0{}_i\,\dot{a}^0{}_j=\begin{cases}\nearrow\ \dfrac{1}{4}\left(1-\dfrac{2}{R^2}T^{\alpha}{}_{\beta}\,\dot{a}_{\alpha 0}\,\dot{a}^{\beta}{}_0\right) & \text{if}\quad i,j=1,2\ \text{or}\ 2,1\\ \rightarrow 0 & \text{otherwise}\\ \searrow\ \dfrac{1}{4}\left(1+\dfrac{2}{R^2}T^{\alpha}{}_{\beta 0}\,\dot{a}_{\alpha 0}\,\dot{a}^{\beta}{}_0\right) & \text{if}\quad i,j=3,4\ \text{or}\ 4,3.\end{cases}$$

But computing $\dfrac{2}{R^2}T^{\alpha}{}_{\beta}\,\dot{a}_{\alpha 0}\,\dot{a}^{\beta}{}_0$ in the frame where $\dot{a}^{\alpha}{}_0=[1,0,0,0]$ one finds at once that

$$\text{(II.1.8.)}\qquad \frac{2}{R^2}T^{\alpha}{}_{\beta}\,\dot{a}_{\alpha 0}\,\dot{a}^{\beta}{}_0=\left(1-\left[\frac{2\vec{E}\times\vec{H}}{E^2+H^2}\right]^2\right)^{-1/2}\geqslant 1.$$

Hence

$$\dot{a}^0{}_1\,\dot{a}^0{}_2\leqslant 0,\qquad \dot{a}^0{}_3{}^0{}_4\geqslant\frac{1}{2},$$

where the equality signs apply when in the frame where $\dot{a}^{\alpha}{}_0=[1,0,0,0]$ the $\vec{E}$ and $\vec{H}$ vectors are parallel. Of course, $\dot{a}^0{}_i{}^{\alpha}$ are eigenvectors of $f^{\alpha}{}_{\beta}$.

Now, suppose that $\frac{2}{R^2} T_{\alpha\beta} \dot{a}^\alpha{}_0 \dot{a}^\beta{}_0 > 1$. In such a case one can rewrite (II.1.5 in the form

(II.1.9)
$$\dot{a}^\alpha(s) = \left[\frac{1}{2}\left(1 + \frac{2}{R} T_{\nu\mu} \dot{a}^\nu{}_0 \dot{a}^\mu{}_0\right)\right]^{-1/2} W^\alpha{}_0 \operatorname{ch}\left[\frac{es}{mc^2} R \sin \Phi\right]$$
$$\left[\frac{1}{2}\left(1 + \frac{2}{R^2} T_{\nu\mu} \dot{a}^\nu{}_0 \dot{a}^\mu{}_0\right)\right]^{-1/2} W^\alpha{}_3 \operatorname{sh}\left[\frac{es}{mc^2} R \sin \Phi\right]$$
$$\left[-\frac{1}{2}\left(1 - \frac{2}{R^2} T_{\nu\mu} \dot{a}^\nu{}_0 \dot{a}^\mu{}_0\right)\right]^{-1/2} W_1{}^\alpha \cos\left[\frac{es}{mc^2} R \cos \Phi\right]$$
$$\left[-\frac{1}{2}\left(1 - \frac{2}{R^2} T_{\nu\mu} \dot{a}^\nu{}_0 \dot{a}^\mu{}_0\right)\right]^{-1/2} W_2{}^\alpha \sin\left[\frac{es}{mc^2} R \cos \Phi\right],$$

where vectors $W^\alpha{}_i$ (their definitions in terms of $\dot{a}^\alpha{}_0$ and $f^\nu{}_\mu$ are obvious by comparing (II.1.9) with (II.1.5)) fulfil

(II.1.10)
$$\eta_{\alpha\beta} W^\alpha{}_i W^\beta{}_j = \eta_{ij}$$

(the last is a direct consequence of (II.1.7)), so that they form an orthonormal "vierbein"

Eqs. (II.1.9) enable us to interpret easily the geometrical meaning of the hodograph of the motion in question, what as completely obvious may be omitted here.

We did use our projection operators in this section in the case of a constant field. It is important, however, to remember that all results of par. 3 in [1] are valid also in the case of a variable field which determines variable fields of our projectors. One can hope that in these cases our projectors may find — because of their remarcable algebraical properties — some less trivial applications.

2. Algebraical properties of $f_{\nu\mu}$ from the point of view of the spinor calculus

Throughout this section the notation of [2] will be adopted.

As it is well known, $f_{\nu\mu}$ is equivalent to a symmetric spinor f_{AB}:

(II.2.1)
$$\begin{cases} f_{\nu\mu} = S_{\nu\mu}{}^{AB} f_{AB} + S_{\nu\mu}{}^{\dot{A}\dot{B}} f_{\dot{A}\dot{B}}, \\ f_{AB} = \frac{1}{8} S_{\nu\mu\, AB} f^{\nu\mu}. \end{cases}$$

The dual tensor is given (because of the self-duality of $S_{\nu\mu}{}^{AB}$) as

(II.2.2)
$$\check{f}^{\nu\mu} = S^{\nu\mu\, AB} f_{AB} - S^{\nu\mu\, \dot{A}\dot{B}} f_{\dot{A}\dot{B}}.$$

Let us investigate algebraical properties of f_{AB}. Define

(II.2.3)
$$f = f_{AB} f^{AB}.$$

Now, using properties of $S_{\nu\mu}{}^{AB}$, one can easily derive that

(II.2.4)
$$f = \frac{1}{4}(F + G), \qquad f^* = \frac{1}{4}(F - G).$$

Moreover, it is easy to prove that

(II.2.5)
$$T^\alpha{}_\beta = 4\, g^{\alpha\, \dot{A}B} g_\beta{}^{\dot{C}D} f_{\dot{A}\dot{C}} f_{BD}.$$

It is obvious that the case of a null-field corresponds to $f = 0$. But $f = 0$ means that the determinant of f_{AB} vanishes; therefore f_{AB} must have the form

$$f_{AB} = \Psi_A \Psi_B. \tag{II.2.6}$$

Hence, the null-field determines one spinor Ψ_A; in accordance with (II.2.5) one can see at once that in such a case

$$T^{\alpha}{}_{\beta} = K^{\alpha} K_{\beta}, \tag{II.2.7}$$

where

$$K^{\alpha} = 2\, g^{\alpha \dot{A} B} \Psi_{\dot{A}} \Psi_B, \qquad K_{\alpha} K^{\alpha} = 0. \tag{II.2.8}$$

Now, assume that $f \neq 0$. The spinor equivalent of the previously studied eigenvalues equation may be written as

$$f^{A}{}_{B} \Psi_i{}^{B} = f_i \Psi_i{}^{A} \tag{II.2.9}$$

which leads easily to the two eigenvalues

$$f_1 = -f', \quad f_2 = f', \tag{II.2.10}$$

where

$$f' = \frac{i}{\sqrt{2}} f^{1/2}$$

it follows at once that f_{AB} may be represented as:

$$f_{AB} = f' (\Psi_{1A} \Psi_{2B} + \Psi_{2A} \Psi_{1B}), \tag{II.2.11}$$

where

$$\Psi_{1A} \Psi_2{}^{A} = 1. \tag{II.2.12}$$

Thus, a not-null field determines a complex scalar f' and two spinors Ψ_{iA} which product is normalized to unity; the spinors Ψ_{iA} are defined with the accuracy up to the transformation

$$\Psi_{1A} \to e^{\lambda} \Psi_{1A}, \qquad \Psi_{2A} \to \bar{e}^{\lambda} \Psi_{2A}. \tag{II.2.13}$$

With the help of our spinors one can construct four null-vectors

$$\begin{cases} u_1{}^{\alpha} = g^{\alpha \dot{A} B} \Psi_{1\dot{A}} \Psi_{1B}, & u_2{}^{\alpha} = g^{\alpha \dot{A} B} \Psi_{2\dot{A}} \Psi_{2C} \\ u_3{}^{\alpha} = g^{\alpha \dot{A} B} \Psi_{1\dot{A}} \Psi_{2B}, & u_4{}^{\alpha} = g^{\alpha \dot{A} B} \Psi_{2\dot{A}} \Psi_{1B}. \end{cases} \tag{II.2.14}$$

Now using the representation of $f_{\nu\mu}$ (II.2.1) and (II.2.11) one can easily check that

$$f^{\alpha}{}_{\beta} u_i{}^{\beta} = \lambda_i u_i{}^{\alpha}, \tag{II.2.15}$$

where λ_i defined in terms of f' as

$$\begin{cases} \lambda_1 = 2(f' - f'^*), & \lambda_2 = 2(-f' + f'^*) \\ \lambda_3 = 2(-f' - f'^*), & \lambda_4 = 2(f' + f'^*), \end{cases} \tag{II.2.16}$$

are eigenvalues of $f^{\alpha}{}_{\beta}$ which were discussed in [1], par. 2, ordered accordingly with (I.2.16).

Remembering (II.2.12) one can check easily that

(II.2.17) $$u_1^\alpha u_{2\alpha} = -u_3^\alpha u_{4\alpha} = \frac{1}{2}, \qquad u_i^\alpha u_{j\alpha} = 0 \quad \text{otherwise}.$$

Of course, $(u_3^\alpha)^* = u_4^\alpha$. Out of four u_i^α, two of them real, remaining two complex, one can construct four real vectors forming a "vierbein".

(II.2.18) $$\begin{cases} w_0^\alpha = u_1^\alpha + u_2^\beta, & w_3^\alpha = u_1^\alpha - u_2^\alpha \\ w_1^\alpha = u_3^\alpha + u_4^\alpha, & w_2^\alpha = i(u_3^\alpha - u_4^\alpha), \end{cases}$$

i.e. w_i^α fulfil:

(II.2.19) $$w_i^\alpha w_{j\alpha} = \eta_{ij}.$$

The transformation (II.2.13) associated with the arbitrariness in the choice of Ψ_i's obviously transforms two blades spaned on u_1^α, u_2^α and u_3^α, u_4^α respectively into themselves. From the point of view of w_i^α vectors it corresponds to a special Lorenz transformation between w_0^α and w_3^α, and to a rotation of w_1^α, w_2^α around w_3^α; it gives a sort of interpretation to two parameters entering in (II.2.13) (λ is here complex).

It does not represent any difficulty to translate all results of par. 2,3 of [1] and par. 1 of this paper into the language of spinors; as a matter of fact these results, after being translated in terms of their spinorial equivalents, are extremely simple and elementary as far as the mechanism of algebraical operations is concerned.

I am indebted to Professor L. Infeld for his valuable suggestions in connection with this work.

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THEORETISCHE PHYSIK

Die magnetischen Strukturen im hexagonalen Gitter

von

W. ZIĘTEK

Vorgelegt von W. RUBINOWICZ am 15. Juni 1961

1. In der Arbeit [1] haben wir ein quantentheoretisches Rechnungsverfahren zur Ableitung der magnetischen Struktur anisotroper ferromagnetischer Einkristalle angegeben und beispielsweise die eindimensionale Lösung für ein Kristall mit einfach kubischem Gitter vorgeführt. In der vorliegenden Arbeit wollen wir die Endresultate für den dreidimensionalen Fall darstellen, zu denen uns die Anwendung unserer Methode auf ein Kristall mit einfach hexagonalem Gitter geführt hat. Dabei haben wir angenommen, dass der Kristall die Gestalt eines Quaders hat, dessen zwei Wände senkrecht zur hexagonalen Gitterachse liegen, die zugleich parallel der Achse x_3 unseres Bezugssystems ist. Das Bezugssystem geht durch die drei Kanten des Quaders (siehe z.B. Abb. 1). Wir haben gleichzeitig die beiden für das hexagonale Gitter grundsätzlichen Strukturen untersucht, nämlich die Struktur

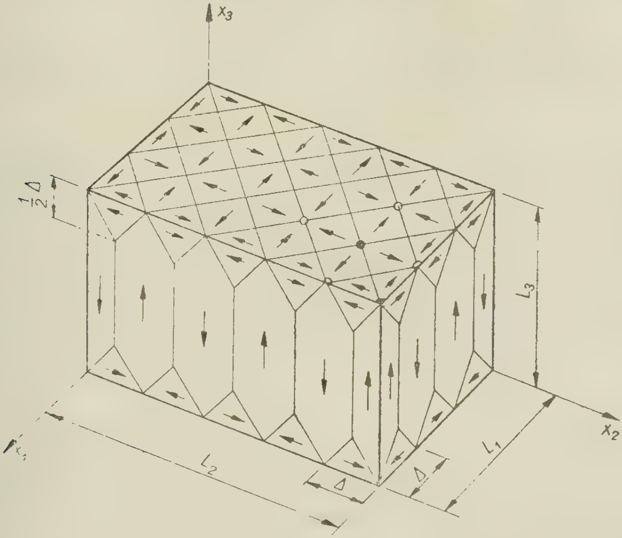

Abb. 1

von Bloch [2], im weiteren kurz SB genannt (siehe Abb. 1), und diejenige von Landau und Lifshitz [3], im folgenden als SLL bezeichnet (siehe Abb. 2). Wie bekannt, ist die erste ein Resultat quantentheoretischer, obwohl sehr grober Erwägungen, dagegen hat sich die zweite auf Grund quasiklassischer Rechnungen als energetisch unbedingt günstiger erwiesen. Insbesondere haben die „inneren" Blochschen Elementarbezirke — wie das ohne weiteres aus Abb. 1 ersichtlich ist — die Form von langgestreckten und beiderseits pyramidenartig zugespitzten Quadern von der Höhe L_3, deren Querdurchschnitt ein Quadrat mit

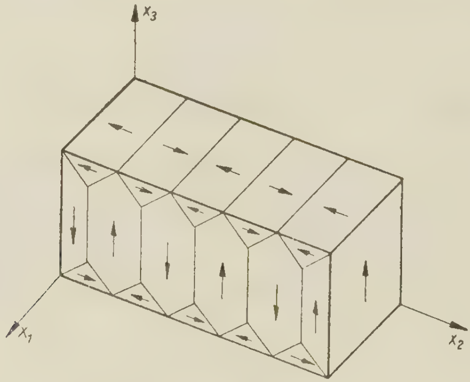

Abb. 2

der Seitenlänge Δ ist (Breite der Elementarbezirke). In der SLL dagegen haben die Elementarbezirke im Innern des Kristalls die Gestalt schichtenförmiger, beiderseits prismenartig abgeschnittener Quadern von der Höhe L_2, der Länge L_1 und der Breite Δ (siehe Abb. 2). Entsprechend unterscheiden sich auch in den beiden Strukturen die sogenannten „zuschliessenden" Bezirke, die als obere und untere Schicht das Einschliessen des Magnetflusses im Kristallinneren bewirken und so das Auftreten von stärkeren örtlichen Magnetfeldern an der Kristalloberfläche verhindern.

Diese beiden Resultate stehen im grundsätzlichen Widerspruch zueinander, den die Erfahrungstatsachen noch bedeutsam vertiefen. Es wurden nämlich experimentel Strukturen ähnelnde beiden festgestellt, jedoch keine von ihnen exakt. Dabei wurden die vom Typus SB öfters beobachtet.

Unser Verfahren erlaubt uns nicht nur diesen Widerspruch zu beseitigen, aber ausserdem führt zu einer Modifizierung beider Strukturen, die mit dem Experiment in viel besserem Einklang ist.

2. Da im hexagonalen Gitter schon die Dipolglieder die Anisotropie bewirken, beschränken wir uns einfachheitshalber zu ihnen und nehmen folgenden Hamiltonoperator an:

$$H = \sum_{\langle\alpha\beta\rangle} P^{\alpha\beta}_{\mu_1 \mu_2} \; S^{\alpha}_{\mu_1} \, S^{\beta}_{\mu_2} \,. \tag{1}$$

Die Bezeichnungen sind grundsätzlich dieselben, wie in [1], nur haben wir hier zu unterscheiden zwischen A_a — dem negativen Austauschintegral für die unmittelbaren Nachbaratome in der zur hexagonalen Gitterachse senkrechten Ebene (wobei a die Gitterkonstante in dieser Ebene bezeichnet), und A_b — dem Austauschintegral für die hexagonale Richtung (wobei b die Gitterkonstante in dieser Richtung ist). Entsprechend haben wir auch zwei positive Koppelungskonstanten C_a und C_b für die Pseudodipolwechselwirkung in diesen beiden Richtungen. Einfachheitshalber lassen wir das äussere Magnetfeld weg. Selbstverständlich ist

(2) $$A_b < A_a < 0, \quad C_b > C_a > 0.$$

3. Im Gegensatz zu [1] bedienen wir uns hier jeweils nur einer unitären Transformation (da das äussere Magnetfeld weggelassen wurde), und zwar:

für die inneren Elementarbezirke

(3) $$U = \exp\left\{i \sum_{\alpha} \varphi^{\alpha} S_m^{\alpha}\right\}, \qquad m = 1 \quad \text{bezw. } 2;$$

für die zuschliessenden Bezirke

(4) $$U = \exp\left\{\frac{i}{\sqrt{2}} \sum_{\alpha} \varphi^{\alpha}(S_3^{\alpha} + S_m^{\alpha})\right\}, \qquad m = 1 \quad \text{bzw. } 2.$$

Nach [1] bedeutet hierbei φ^{α} den Drehwinkel des Magnetisierungsvektors beim Übergang vom Atom zum Atom im Gitter.

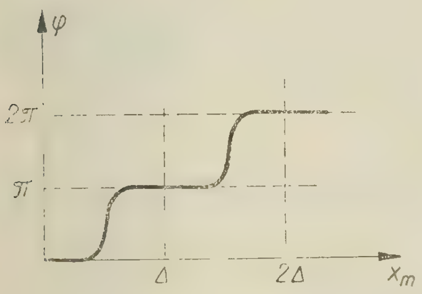

Abb. 3

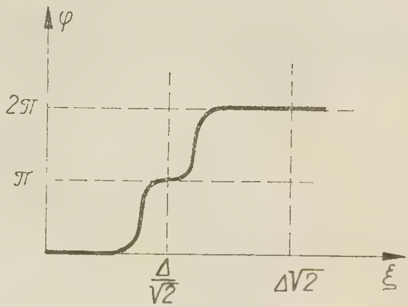

Abb. 4

Nachdem wir an (1) die Transformationen (3) und (4) entsprechend ausüben, bestimmte Näherungen machen und das in [1] angedeutete Variationsminimalisierungsverfahren anwenden, bekommen wir folgende Lösungen für φ:

für die inneren Bezirke

(5) $$\cos\varphi = -\operatorname{sn}\left(2K_0 \frac{x_m}{\Delta} - K_0\right), \qquad K_0 = K(k_0), \quad k_0 \lesssim 1;$$

für die zuschliessenden Bezirke

(6) $$\cos\varphi = \frac{k^2 - \operatorname{sn} t}{1 - k^2 \operatorname{sn} t}, \qquad t = \frac{2K}{\Delta}(x_3 + x_m) - K, \quad k \lesssim 1.$$

Hier bedeuten: k_0, k — Moduls der entsprechenden elliptischen Funktionen; K_0, K — volle elliptische Integrale erster Gattung. Überdies $m = 1$ bzw. 2. Beide Lösungen sind in Abb. 3 und 4 dargestellt, wobei in Abb. 4 für die Lösung (6) die Abszisse ξ der Geraden $x_3 = x_m$ entspricht, die senkrecht die Blochwand in der zuschliessenden Schicht durchsticht.

Aus (5) und (6) folgt unmittelbar die „modifizierte" SB (siehe Abb. 5) und SLL (siehe Abb. 6). Man bekommt sie auch, wenn man in Abb. 1 und 2 einen Teil der zuschliessenden Schichten abschneidet.

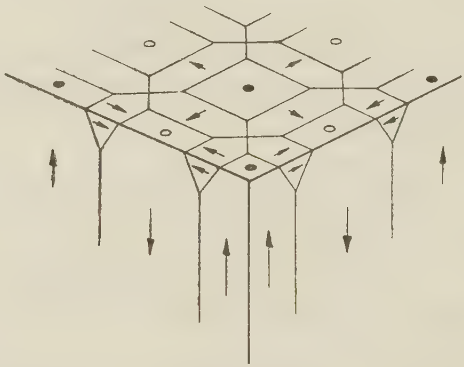

Abb. 5

Es ist ersichtlich, dass in diesem Fall der Magnetfluss nicht im Kristallinneren eingeschlossen ist und an der hexagonalen Kristalloberfläche örtliche ziemlich starke Magnetfelder zu vermuten sind. Eben das bestetigen die bekannten Pulverfi-

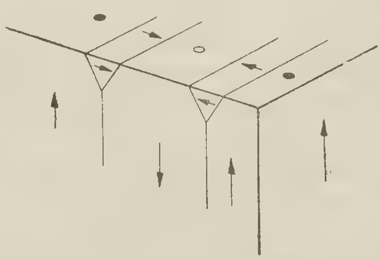

Abb. 6

gurenversuche an Co von Bitter [4] und Williams [5], und noch besser die Versuche von Germer [6], der sogar die Stärke der örtlichen Magnetfelder gemessen hat. die ungefähr 10^4 Oe betrug!

4. Man kann noch sehr einfach die Energien der modifizierten SB und SLL berechnen und vergleichen und damit auf Grund quantentheoretischer Betrachtung entscheiden, welche von ihnen energetisch günstiger ist. Die zusätzliche Minimalisierung der mittleren Energie h_0 (siehe [1]) in Bezug auf die Breite Δ der Elemen-

tarbezirke erlaubt diese zu bestimmen [3] und somit das Minimum von h_0 in gewisser Näherung auszurechnen. Für die modifizierte SB bekommt man sodann

$$h_0^{\min} = -E_0 - E_0' + E_0'' \sqrt{\varepsilon}, \tag{7}$$

und für die modifizierte SLL

$$\tilde{h}_0^{\min} = -E_0 - \frac{1}{2} E_0' + E_0'' \sqrt{\frac{\varepsilon'}{2}}, \tag{8}$$

wo

$$\begin{cases} E_0 = -3 NS^2 (A_a + C_a) - 2 NS^2 (A_b - 2 C_b) > 0, \\ E_0' = 6 S^2 L_1 L_2 a V_0^{-1} \sqrt{2 (4 A_a + C_a) (3 C_a - 4 C_b)} > 0, \\ E_0'' = 6 S^2 L_1 L_2 V_0^{-1} \sqrt{3 a L_3} \sqrt[4]{2 (4 A_a + C_a) (3 C_a - 4 C_b)^3} > 0, \end{cases} \tag{9}$$

$$V_0 = \frac{3}{2} \sqrt{3}\, a^2 b, \quad 0 < \frac{1}{2} \varepsilon < \varepsilon' < \varepsilon \ll 1 .$$

Weitere Bezeichnungen — siehe [1]. In der Arbeit [3] hat man nur die dritten Glieder in [7] und [8] berechnet und verglichen und daraus den dann schon folgerichtigen Schluss gezogen, dass die SLL — allerdings die aus Abb. 2 — immer energetisch günstiger ist, als die SB. Aus unseren Rechnungen sieht man deutlich. dass das nicht immer der Fall ist, vielmehr von L_3 abhängt, d.h. von der Kristalldicke in der hexagonalen Richtung. Aus der Bedingung

$$h_0^{\min} - \tilde{h}_0^{\min} = 0 \tag{10}$$

kann man leicht die kritische Dicke L_3^0 bestimmen, und zwar

$$L_3^0 = \frac{1}{3} a (2 \sqrt{\varepsilon} - \sqrt{2 \varepsilon'})^{-2} \{2 (4 A_a + C_a)/(3 C_a - 4 C_b)\}^{1/2}. \tag{11}$$

Alsdann haben wir für $L_3 > L_3^0$ die modifizierte SLL aus Abb. 6, dagegen für $L_3 < L_3^0$ die modifizierte SB aus Abb. 5, was in guter qualitativer Übereinstimmung mit der Erfahrung ist. Die SB wurde in den bisherigen Versuchen daher öfters beobachtet, da man es gewiss meistens mit Einkristallen unter der kritischen Dicke zu tun hatte.

Ein quantitativer Vergleich mit dem Experiment ist leider noch kaum möglich, da einerseits die in (11) vorkommenden Grössen nicht alle genug genau bekannt sind und andererseits fehlt es noch an entsprechenden systematischen Versuchen. Eine grobe Abschätzung mittels bisher bekannter Werte für die in (11) auftretenden Konstanten ergibt nämlich $L_3 \sim 10^4\, a$, was allen Anscheins nach mindestens um zwei Grössenordnungen zu wenig ist.

Ein ausführlicher Bericht über die hier angegebenen Resultate wird in Acta Physica Polonica erscheinen.

Herrn Prof. Dr. R. S. Ingarden danke ich für werten Meinungsaustausch hinsichtlich der Arbeit.

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ANMERKUNG: In der Arbeit [1] enthält die Gleichung (10) einen Druckfehler: ihre korekte Form ist folgende: $\sin 2\varphi = \mathrm{sn}\xi$.

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THEORETICAL PHYSICS

On the Wave Function of the Anomalous State of Fermi-system

by

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1. Introduction

In our previous papers [1], [2] the extremal case of an anisotropic conductor leading to the anomalous state was considered. In this state the correlations of pairs of particles with parallel spins are favoured. These correlations are the consequence of a special type of interaction between the particles. In the expansion of the interaction energy there must occur the spherical harmonics with odd indices only. In [1]—[2] a case of the axial symmetry was considered. The axis of symmetry was choosen as quantization axis of the electron spins. In this paper we shall deal with the ground state and the excited states of the anomalous state.

2. Ground state

Consider a dynamical system of Fermi-particles with a Hamiltonian

$$(1)\quad \hat{H} \quad \sum_{\vec{k}_1\sigma} (\varepsilon(k) - \lambda)\, a^{+}_{\sigma k}\, a_{k\sigma} + \frac{1}{2} \sum_{\substack{\vec{k}_1+\vec{k}_2=\vec{k}_1{}' \quad k_2{}',\\ \sigma}} J(\vec{k}_1, \vec{k}_2; \vec{k}_2', \vec{k}_1')\, a^{+}_{k_1\sigma_1} a^{+}_{k_2\sigma_2} a_{k_2'\sigma_2} a_{k_1'\sigma_1},$$

where $a_{k\sigma}$, $a^{+}_{k\sigma}$ are the Fermi-amplitudes, $\vec{k}$, σ — wave vector and spin index, respectively, (in the notation of $a_{k\sigma}$, $a^{+}_{k\sigma}$ we omit the arrow upon vector $\vec{k}$), $\varepsilon(k)$ — the energy of the one-particle state $|k, \sigma\rangle$, and λ — chemical potential.

The vacuum state in a-representation is defined by

$$(2)\qquad a\psi = 0.$$

Repeating the procedure of [1] we pass to new Fermi-amplitudes α, α^{+}

$$(3a)\qquad a_{k\sigma} = u(\vec{k})\, \alpha_{k\sigma} - v(\vec{k}, \sigma)\, \alpha^{+}_{-k\sigma},$$

$$(3b)\qquad a_{k\sigma} = u(\vec{k})\, \alpha_{k\sigma} + v(\vec{k}, \sigma)\, \alpha^{+}_{-k\sigma}.$$

The functions $\{u, v\}$ depend on the direction of the vector $\vec{k}$ and, moreover, satisfy the relations

$$(4)\qquad \begin{cases} u(\vec{k}) = u(-\vec{k}), & v(\vec{k}_1\sigma) = -v(-\vec{k}_1\sigma), \\ v(\vec{k}_1\sigma) = -v(\vec{k}_1 -\sigma), & v(\vec{k}_1+) = v(\vec{k}). \end{cases}$$

We shall prove, using the method similar to that developed by Yosida [3] for the superconducting state, that the transformation (3b) can be obtained by unitary operator $\exp S$

$$e^S a_{ps} e^{-S} = u(\vec{p})\, a_{ps} + v(\vec{p}, s)\, a^+_{-ps} = \alpha_{ps}, \tag{5}$$

where

$$S = \sum_{\vec{k}, \sigma} \frac{\Theta_{k\sigma}}{2} \left(a^+_{k\sigma} a^+_{-k\sigma} - a_{-k\sigma} a_{k\sigma}\right) \tag{6}$$

and

$$\begin{cases} \Theta_{k\sigma} = -\Theta_{-k,\sigma}, \quad \Theta_{k\sigma} = -\Theta_{k,-\sigma}, \quad \Theta_{k+} = \Theta_k, \\ \cos\Theta_k = u(\vec{k}), \quad \sin\Theta_{k\sigma} = v(\vec{k}, \sigma), \end{cases} \tag{7}$$

(the functions $\Theta_{k\sigma}$ depend on the direction of the vector $\vec{k}$, but we omit in the notation the arrow upon $\vec{k}$).

Deriving (5) we make use of the equation

$$e^S A e^{-S} = A + \frac{1}{1!}[S, A] + \frac{1}{2!}[S, A]] + + = \sum_{n=0}^{\infty} \frac{1}{n!} \overset{(n)}{[S, A]}. \tag{8}$$

In our case

$$\overset{(2n)}{[S, a_{k\sigma}]} = (-1)^n \Theta^{2n}_{k\sigma} a_{k\sigma}, \quad \overset{(2n+1)}{[S, a_{k\sigma}]} = (-1)^n \Theta^{2n+1}_{k\sigma} a^+_{-k\sigma}. \tag{9}$$

Formula (2) may be written as follows

$$e^S a e^{-S} e^S \psi = \alpha\, C_a = 0, \tag{10}$$

where C_a is the vacuum state for quasiparticles in α-representation

$$C_a = e^S \psi = \prod_{\substack{\vec{k}, \sigma \\ kz>0}} [u(\vec{k}) - v(\vec{k}, \sigma)\, a^+_{k\sigma} a^+_{-k\sigma}]\, \psi = \tag{11}$$

$$= \prod_{\vec{k}, kz>0} [u(\vec{k})^2 - u(\vec{k})\, v(\vec{k})\, (a^+_{k+} a^+_{-k-} - a^+_{k-} a^+_{-k+}) + v^2(\vec{k})\, a^+_{k+} a^+_{-k+} a^+_{k-} a^+_{-k-}]\, \psi,$$

(the function of this form was proposed as a trial wave function in [4]). We see that C_a is the S-state function for it does not depend on the direction. Such being the case, C_a is actually the ground state function.

3. Excited states

Let us consider now the one-particle excited states $a^+_{ps} C_a$

$$\begin{cases} a^+_{+p+} C_a = \prod\limits_{\substack{\vec{k} \neq \vec{p}, \sigma \\ kz>0}} (u(\vec{k}) - v(\vec{k}, \sigma)\, a^+_{k\sigma} a^+_{-k\sigma})\, (u(\vec{p}) - v(\vec{p})\, a^+_{p-} a^+_{-p-})\, a^+_{+p+} \psi, \\ a^+_{+p-} C_a = \prod\limits_{\substack{\vec{k} \neq \vec{p}, \sigma \\ kz>0}} (u(\vec{k}) - v(\vec{k}, \sigma)\, a^+_{k\sigma} a^+_{-k\sigma})\, (u(\vec{p}) - v(\vec{p})\, a^+_{p+} a^+_{-p+})\, a^+_{+p-} \psi. \end{cases} \tag{12}$$

In the last expressions, the factors depending on $u(\vec{p})$, $v(\vec{p}, \sigma)$, (i.e. on the direction of the vector $\vec{p}$), are subtracted. Consequently, the wave functions (12) depend on the direction of $\vec{p}$. Thus (12) are the P-state functions.

The two-particle states are

$$a^+_{p-} a^+_{l-} C_a = \prod (u(\vec{k}) - v(\vec{k}, \sigma) a^+_{k\sigma} a^+_{-k\sigma}) (u(\vec{p}) - v(\vec{p}) a^+_{p+} a^+_{-p+}) \times$$
$$\times (u(\vec{l}) - v(\vec{l}) a^+_{l+} a^+_{-l+}) a^+_{p-} a^+_{l-} \psi ,$$

(13) $$a^+_{p+} a^+_{l-} C_a = \prod (u(\vec{k}) - v(\vec{k}, \sigma) a^+_{k\sigma} a^+_{-k\sigma}) (u(\vec{p}) + v(\vec{p}) a^+_{p-} a^+_{-p-}) \times$$
$$\times (u(\vec{l}) - v(\vec{l}) a^+_{l+} a^+_{-l+}) a^+_{p+} a^+_{l-} \psi ,$$

$$a^+_{p+} a^+_{l+} C_a = \prod_{\substack{\vec{k} \neq \vec{p} \\ \vec{k} \neq \vec{l}, \sigma \\ kz > 0}} (u(\vec{k}) - v(\vec{k}, \sigma) a^+_{k\sigma} a^+_{-k\sigma}) (u(\vec{p}) + v(\vec{p}) a^+_{p-} a^+_{-p-}) \times$$
$$\times (u(\vec{l}) + v(\vec{l}) a^+_{l-} a^+_{-l-}) a^+_{p+} a^+_{l+} \psi .$$

For $\vec{l} = -\vec{p}$ we have

$$a^+_{p-} a^+_{-p-} C_a = \prod_{\substack{\vec{k} \neq \vec{p}, \sigma \\ kz > 0}} (u(\vec{k}) - v(\vec{k}, \sigma) a^+_{k\sigma} a^+_{-k\sigma}) (u(\vec{p}) - v(\vec{p}) a^+_{p+} a^+_{-p+}) \times$$
$$\times (u(\vec{p}) a^+_{p-} a^+_{-p-} - v(\vec{p})) \psi ,$$

(14) $$a^+_{p+} a^+_{-p-} C_a = \prod (u(\vec{k}) - v(\vec{k}, \sigma) a^+_{k\sigma} a^+_{-k\sigma}) a^+_{p+} a^+_{-p-} \psi ,$$

$$a^+_{p+} a^+_{-p+} C_a = \prod (u(\vec{k}) - v(\vec{k}, \sigma) a^+_{k\sigma} a^+_{-k\sigma}) (u(\vec{p}) + v(\vec{p}) a^+_{p-} a^+_{-p-}) \times$$
$$\times (u(\vec{p}) a^+_{p+} a^+_{-p+} + v(\vec{p})) \psi .$$

which means that we have a triplet of the P-state with the spins $-1, 0, 1$.

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Space Group of White Tin. IV. Basis Functions for the Irreducible Representations at Symmetry Points

by

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The wide experimental interest in the electronic properties of white tin [1]—[5] suggests to investigate the electronic band structure of this metal. A group-theoretical analysis of the space group of white tin [6]—[9] allows to find the basis functions which transform according to the irreducible representations of the group of the wave vector at symmetry points of the Brillouin zone.

Following the work already done for diamond [10]—[14], zinc blende [15], [16] and wurtzite [17] we give here the symmetrized plane waves of low kinetic energy at five symmetry points of the Brillouin zone of white tin.

In the orthogonalized plane wave method an expansion for the wave function at the point $\boldsymbol{k}$ has the form

(1) $$\Psi_{\varkappa}^{(j)}(\boldsymbol{r}, \boldsymbol{k}) = \sum_{n} a_n \psi_{\varkappa}^{(j)}(\boldsymbol{r}, \boldsymbol{k}+\boldsymbol{k}_n).$$

The plane waves included in the summation (1) are symmetrized, i.e. each one transforms according to a specified row of a particular irreducible representation $D^{(j)}(S)_{\varkappa\lambda}$.

A symmetrized plane wave is explicitly constructed by exploiting the rule given by Wigner [18], [19]:

(2) $$\psi_{\varkappa}^{(j)}(\boldsymbol{r}, \boldsymbol{k}+\boldsymbol{k}_n) = \sum_{S} D^{(j)}(S)_{\varkappa\varkappa}^{*} \mathbf{P}_S \exp[i(\boldsymbol{k}+\boldsymbol{k}_n)\boldsymbol{r}].$$

The summation here goes over all space operations S the rotational parts of which form the group of the wave vector $\boldsymbol{k}$.

The vectors $\boldsymbol{k}+\boldsymbol{k}_n$ in (i) are formed by adding to $\boldsymbol{k}$ any primitive vectors $\boldsymbol{k}_n$ of the reciprocal lattice. In the body-centered tetragonal lattice with lattice constants a and c we choose the basic primitive vectors in the reciprocal space

(3) $$\boldsymbol{b}_1 = 2\pi(0, 1/a, 1/c), \quad \boldsymbol{b}_2 = 2\pi(1/a, 0, 1/c), \quad \boldsymbol{b}_3 = 2\pi(1/a, 1/a, 0).$$

A primitive vector in $\boldsymbol{k}$ space $m_1\boldsymbol{b}_1+m_2\boldsymbol{b}_2+m_3\boldsymbol{b}_3$, with m_1, m_2, m_3 any integers, can be written in Cartesian co-ordinates $\boldsymbol{k}_n = 2\pi(n_1/a, n_2/a, n_3/c)$, where

(4) $$n_1 = m_2+m_3, \quad n_2 = m_1+m_3, \quad n_3 = m_1+m_2.$$

Thus the sum $n_1+n_2+n_3$ has to be even.

The wave vector $\boldsymbol{k}+\boldsymbol{k}_n$ may then be written as

$$\boldsymbol{k}+\boldsymbol{k}_n = \pi\,[(2n_1+s_1)/a, \quad (2n_2+s_2)/a, \quad (2n_3+s_3)/c]. \tag{5}$$

The numbers $(\pi s_1/a,\ \pi s_2/a,\ \pi s_3/c)$ for the five symmetry points $\boldsymbol{k}$ are given at the headline of each Table. In order to simplify the labelling of plane waves [19] the triples of numbers in the first column in each Table stand simply for $2n_1+s_1$, $2n_2+s_2$, $2n_3+s_3$. Under the operations of the group of the vector $\boldsymbol{k}$ these numbers are permuted among themselves with possible change of sign. We write $\dot{n}_l$ for $-n_l$. The set of numbers obtained by the application of the operations of the $\boldsymbol{k}$-vector group gives one plane wave type.

In some cases a given irreducible representation occurs more then once, for instance f times, in a given plane wave type [19]. Then it is possible to construct f linearly independent wave functions of given symmetry. In such cases we give in the Tables the coefficients of all f functions in f columns separated by a space. The instances with $f=2$ happen for the types: at Γ for (422) and for (620), at X for (310), and at P for (311).

To write down the actual linear combination which transforms according to a specified row of the irreducible representation one has only to add the waves belonging to the given plane wave type with coefficients given in the Table. A few explicit examples of unnormalized symmetrized plane waves are given under Tables I and IV. Each combination may be normalized over a unit cell of volume Ω by dividing by $(\Omega\,\Sigma)^{1/2}$, where Σ is the sum of the absolute squares of the coefficients given in the Table.

We have to mention that in parts I, II, III [7]—[9] we did not use the space operations in the sense as is used here, namely $P_S\,\psi(\boldsymbol{r}) = \psi(S^{-1}\boldsymbol{r})$, but instead we used $P_S\,\psi(\boldsymbol{r}) = \psi(S\boldsymbol{r})$. This implied a change of the representations at the point P. The matrices of the representations P_1 and P_2 which we use here are complex conjugates of the matrices P_1 and P_2 respectively listed in I.

For the centre of the zone we give further the basis functions of the irreducible representations of the double group in the tight-binding limit. We give the spinor

$$\psi_\varkappa(\boldsymbol{r}) = \begin{bmatrix} u_\varkappa(\boldsymbol{r}) \\ v_\varkappa(\boldsymbol{r}) \end{bmatrix}, \tag{6}$$

which transforms under a space operation S according to the $\varkappa$-th row of the irreducible d-dimensional representation $D(S_{\varkappa\lambda})$. The operator $\boldsymbol{O}_S$, operating on the spinor wave function, is a product $\boldsymbol{O}_S = P_S\,\boldsymbol{Q}_S$ of two operators the first of which acts on the Cartesian coordinates and the second operates on spin coordinates:

$$\boldsymbol{O}_S\,\psi_\lambda(\boldsymbol{r}) = P_S\,\boldsymbol{Q}_S\,\psi_\lambda(\boldsymbol{r}) = D^{(1/2)}(S)\begin{bmatrix} u_\lambda(S^{-1}\boldsymbol{r}) \\ v_\varkappa(S^{-1}\boldsymbol{r}) \end{bmatrix} = \sum_{\varkappa=1}^{d} D(S)_{\varkappa\lambda}\,\psi_\varkappa(\boldsymbol{r}). \tag{7}$$

For the matrix S of the transformation of Cartesian coordinates and for the two-dimensional spin matrix $D^{(1/2)}(S)$ we use the convention of Lomont [20].

TABLE I

The point Γ (0, 0, 0).

/a, π/a, π/c	Γ_1^+	Γ_2^+	Γ_3^+	Γ_4^+	Γ_5^+ 1	Γ_5^+ 2	Γ_1^-	Γ_2^-	Γ_3^-	Γ_4^-	Γ_5^- 1	Γ_5^- 2
0 0 0	+	0	0	0	0	0	0	0	0	0	0	0
2 2 0	0	0	0	0	+	+	+	0	0	+	0	0
2 $\bar{2}$ 0	0	0	0	0	+	—	+	0	0	—	0	0
$\bar{2}$ 2 0	0	0	0	0	—	+	+	0	0	—	0	0
$\bar{2}$ $\bar{2}$ 0	0	0	0	0	—	—	+	0	0	+	0	0
4 0 0	+	0	+	0	0	0	0	0	0	0	+	0
$\bar{4}$ 0 0	+	0	+	0	0	0	0	0	0	0	—	0
0 4 0	+	0	—	0	0	0	0	0	0	0	0	+
0 $\bar{4}$ 0	+	0	—	0	0	0	0	0	0	0	0	—
2 0 2	+	0	+	0	+	0	+	0	+	0	+	0
2 0 $\bar{2}$	—i	0	—i	0	i	0	i	0	i	0	—i	0
$\bar{2}$ 0 2	+	0	+	0	—	0	+	0	+	0	—	0
$\bar{2}$ 0 $\bar{2}$	—i	0	—i	0	—i	0	i	0	i	0	i	0
0 2 2	—i	0	i	0	0	+	i	0	—i	0	0	+
0 2 $\bar{2}$	+	0	—	0	0	i	+	0	—	0	0	—i
0 $\bar{2}$ 2	—i	0	i	0	0	—	i	0	—i	0	0	—
0 $\bar{2}$ $\bar{2}$	+	0	—	0	0	—i	+	0	—	0	0	i
4 2 2	+	+	+	+	+ 0	+ 0	+	+	+	+	+ 0	+ 0
4 2 $\bar{2}$	i	i	i	i	—i 0	—i 0	—i	—i	—i	—i	i 0	i 0
4 $\bar{2}$ 2	+	—	+	—	+ 0	— 0	+	—	+	—	+ 0	— 0
4 $\bar{2}$ $\bar{2}$	i	—i	i	—i	—i 0	i 0	—i	i	—i	i	i 0	—i 0
$\bar{4}$ 2 2	+	—	+	—	— 0	+ 0	+	—	+	—	— 0	+ 0
$\bar{4}$ 2 $\bar{2}$	i	—i	i	—i	i 0	—i 0	—i	i	—i	i	—i 0	i 0
$\bar{4}$ $\bar{2}$ 2	+	+	+	+	— 0	— 0	+	+	+	+	— 0	— 0
$\bar{4}$ $\bar{2}$ $\bar{2}$	i	i	i	i	i 0	i 0	—i	—i	—i	—i	—i 0	—i 0
2 4 2	i	—i	—i	i	0 +	0 +	—i	i	i	—i	0 +	0 +
2 4 $\bar{2}$	+	—	—	+	0 —i	0 —i	+	—	—	+	0 i	0 i
$\bar{2}$ 4 2	i	i	—i	—i	0 —	0 +	—i	—i	i	i	0 —	0 +
$\bar{2}$ 4 $\bar{2}$	+	+	—	—	0 i	0 —i	+	+	—	—	0 —i	0 i
2 $\bar{4}$ 2	i	i	—i	—i	0 +	0 —	—i	—i	i	i	0 +	0 —
2 $\bar{4}$ $\bar{2}$	+	+	—	—	0 —i	0 i	+	+	—	—	0 i	0 —i
$\bar{2}$ $\bar{4}$ 2	i	—i	—i	i	0 —	0 —	—i	i	i	—i	0 —	0 —
$\bar{2}$ $\bar{4}$ $\bar{2}$	+	—	—	+	0 i	0 i	+	—	—	+	0 —i	0 —i
4 4 0	+	0	0	+	0	0	0	0	0	0	+	+
4 $\bar{4}$ 0	+	0	0	+	0	0	0	0	0	0	+	—
$\bar{4}$ 4 0	+	0	0	—	0	0	0	0	0	0	—	+
$\bar{4}$ $\bar{4}$ 0	+	0	0	—	0	0	0	0	0	0	—	—

Table I. (continued)

$\pi/a,\ \pi/a,\ \pi/c$	Γ_1^+	Γ_2^+	Γ_3^+	Γ_4^+	Γ_5^+		Γ_1^-	Γ_2^-	Γ_3^-	Γ_4^-	Γ_5^-	
					1	2					4	2
$6\ 2\ 0$	0	0	0	0	+ 0	+ 0	+	+	+	+	0	0
$6\ \bar{2}\ 0$	0	0	0	0	+ 0	− 0	+	−	+	−	0	0
$\bar{6}\ 2\ 0$	0	0	0	0	− 0	+ 0	+	−	+	−	0	0
$\bar{6}\ \bar{2}\ 0$	0	0	0	0	− 0	− 0	+	+	+	+	0	0
$2\ 6\ 0$	0	0	0	0	0 +	0 +	+	−	−	+	0	0
$\bar{2}\ 6\ 0$	0	0	0	0	0 −	0 +	+	+	−	−	0	0
$2\ \bar{6}\ 0$	0	0	0	0	0 +	0 −	+	+	−	−	0	0
$\bar{2}\ \bar{6}\ 0$	0	0	0	0	0 −	0 −	+	−	−	+	0	0
$6\ 0\ 2$	+	0	+	0	+	0	+	0	+	0	+	0
$6\ 0\ \bar{2}$	$-i$	0	$-i$	0	i	0	i	0	i	0	$-i$	0
$\bar{6}\ 0\ 2$	+	0	+	0	−	0	+	0	+	0	−	0
$\bar{6}\ 0\ \bar{2}$	$-i$	0	$-i$	0	$-i$	0	i	0	i	0	i	0
$0\ 6\ \bar{2}$	+	0	−	0	0	i	+	0	−	0	0	$-i$
$0\ 6\ 2$	$-i$	0	i	0	0	−	i	0	$-i$	0	0	−
$0\ \bar{6}\ 2$	+	0	−	0	0	$-i$	+	0	−	0	0	i
$0\ 6\ 2$	$-i$	0	i	0	0	+	i	0	$-i$	0	0	+
$0\ 0\ 4$	0	0	+	0	0	0	+	0	0	0	0	0
$0\ 0\ 4$	0	0	−	0	0	0	+	0	0	0	0	0

The functions of the symmetry type Γ_5^+ for the set (220) are: $\psi_1 = 4i \sin 2\pi x/a \cos 2\pi y/a$, $\psi_2 = 4i \sin 2\pi y/a \cos 2\pi x/a$.

The functions of the symmetry type Γ_5^- for the set (400) are: $\psi_1 = 2i \sin 4\pi x/a$, $\psi_2 = 2i \sin 4\pi y/a$.

TABLE II

The point $L\,(2\pi/a,\ 0,\ 0)$

$\pi/a,\ \pi/a,\ \pi/c$	L_1		L_2		L_3		L_4	
	1	1	1	2	1	2	1	2
$2\ 0\ 0$	0	0	+	+	+	0	0	0
$\bar{2}\ 0\ 0$	0	0	+	+	−	0	0	0
$0\ \bar{2}\ 0$	0	0	+	−	0	−	0	0
$0\ 2\ 0$	0	0	+	−	0	+	0	0
$0\ 0\ 2$	0	0	+	+	0	0	0	0
$0\ 0\ 2$	0	0	+	−	0	0	0	0
$4\ 2\ 0$	+	+	+	+	0	+	+	0
$4\ \bar{2}\ 0$	−	−	+	+	0	−	+	0
$4\ 2\ 0$	−	−	+	+	0	+	−	0

Table II. (continued)

$\pi/a, \pi/a, \pi/c$	L_1		L_2		L_3		L_4	
	1	2	1	2	1	2	1	2
$\bar{4}$ $\bar{2}$ 0	+	+	+	+	0	—	—	0
2 4 0	—	+	+	—	+	0	0	+
2 $\bar{4}$ 0	+	—	+	—	+	0	0	—
$\bar{2}$ 4 0	+	—	+	—	—	0	0	+
$\bar{2}$ $\bar{4}$ 0	—	+	+	—	—	0	0	—
2 2 2	+	+	+	+	+	+	+	+
2 2 $\bar{2}$	—	+	+	—	i	$-i$	$-i$	i
2 $\bar{2}$ 2	—	—	+	+	+	—	+	—
2 $\bar{2}$ $\bar{2}$	+	—	+	—	i	i	$-i$	$-i$
$\bar{2}$ 2 2	—	—	+	+	—	+	—	+
$\bar{2}$ 2 $\bar{2}$	+	—	+	—	$-i$	$-i$	i	i
$\bar{2}$ $\bar{2}$ 2	+	+	+	+	—	—	—	—
$\bar{2}$ $\bar{2}$ $\bar{2}$	—	+	+	—	$-i$	i	i	$-i$

TABLE III

The point $X(\pi/a, \pi/a, 0)$

$\pi/a, \pi/a, \pi/c$	X_1		X_2	
	1	2	1	2
1 1 0	+	+	0	0
$\bar{1}$ $\bar{1}$ 0	+	—	0	0
3 $\bar{1}$ 0	+	+	+	+
$\bar{3}$ 1 0	+	—	+	—
1 $\bar{3}$ 0	+	—	—	+
$\bar{1}$ 3 0	+	+	—	—
1 $\bar{1}$ 2	+	+	+	+
$\bar{1}$ 1 2	+	—	+	—
1 $\bar{1}$ $\bar{2}$	+	—	—	+
$\bar{1}$ 1 $\bar{2}$	+	+	—	—
3 1 2	+ 0	+ 0	+ 0	+ 0
$\bar{3}$ $\bar{1}$ 2	+ 0	— 0	+ 0	— 0
1 $\bar{3}$ 2	+ 0	— 0	— 0	- 0
1 3 $\bar{2}$	+ 0	+ 0	— 0	— 0
3 1 $\bar{2}$	0 +	0 +	0 +	0 +
$\bar{3}$ $\bar{1}$ $\bar{2}$	0 +	0 —	0 +	0 —
$\bar{1}$ 3 2	0 +	0 —	0 —	0 +
1 3 2	0 +	0 +	0 —	0 —

TABLE IV

The point $P(\pi/a,\ \pi/a,\ \pi/c)$

$\pi/a,\ \pi/a,\ \pi/c$	P_1 1	P_1 2	P_2 1	P_2 2
$1\ 1\ 1$	$+$	$+$	$+$	$+$
$\bar{1}\ \bar{1}\ 1$	$+$	$-$	$+$	$-$
$\bar{1}\ 1\ \bar{1}$	$+$	$-i$	$-$	i
$1\ \bar{1}\ \bar{1}$	$+$	i	$-$	$-i$
$3\ 1\ 1$	$+\ \ 0$	$+\ \ 0$	$+\ \ 0$	$+\ \ 0$
$\bar{3}\ \bar{1}\ \bar{1}$	$+\ \ 0$	$-\ \ 0$	$+\ \ 0$	$-\ \ 0$
$\bar{1}\ 3\ 1$	$+\ \ 0$	$-i\ \ 0$	$-\ \ 0$	$i\ \ 0$
$1\ \bar{3}\ 1$	$+\ \ 0$	$i\ \ 0$	$-\ \ 0$	$-i\ \ 0$
$3\ \bar{1}\ 1$	$0\ \ +$	$0\ \ +$	$0\ \ +$	$0\ \ +$
$3\ 1\ 1$	$0\ \ +$	$0\ \ -$	$0\ \ +$	$0\ \ -$
$1\ 3\ 1$	$0\ \ +$	$0\ \ -i$	$0\ \ -$	$0\ \ i$
$\bar{1}\ \bar{3}\ \bar{1}$	$0\ \ +$	$0\ \ i$	$0\ \ -$	$0\ \ -i$

The functions of the symmetry type P_1 for the set (111) are:

$$\psi_1 = 4(\cos \pi x/a \cos \pi y/a \cos \pi z/c - i \sin \pi x/a \sin \pi y/a \sin \pi z/c),$$

$$\psi_2 = -2\,[\sin \pi z/c \sin \pi (x+y)/a + \cos \pi z/c \sin \pi (x-y)/a] + 2i\,[\cos \pi z/a \sin \pi (x+y)/a + \sin \pi z/c \sin \pi (x-y)/a].$$

The functions of the symmetry type P_2 for the set (111) are:

$$\psi_1 = 4(-\sin \pi x/a \sin \pi y/a \cos \pi z/c + i \cos \pi x/a \cos \pi y/a \sin \pi z/c),$$

$$\psi_2 = -2\,[\sin \pi z/c \sin \pi (x+y)/a - \cos \pi z/c \sin \pi (x-y)/a] + 2i\,[\cos \pi z/c \sin \pi (x+y)/a - \sin \pi z/c \sin \pi (x-y)/a].$$

TABLE V

The point $V(\pi/a,\ 0,\ \pi/c)$. Here $\varepsilon = \exp(i\pi/4)$

$\pi/a,\ \pi/a,\ \pi/c$	V_1	V_2	V_3	V_4
$1\ 0\ 1$	$+$	$+$	0	0
$\bar{1}\ 0\ \bar{1}$	$-\varepsilon^*$	ε^*	0	0
$1\ 2\ \bar{1}$	$+$	$+$	$+$	$+$
$1\ \bar{2}\ \bar{1}$	$+$	$+$	$-$	$-$
$1\ \bar{2}\ 1$	ε	$-\varepsilon$	ε	$-\varepsilon$
$\bar{1}\ 2\ 1$	ε	$-\varepsilon$	$-\varepsilon$	ε

TABLE VI

Basis functions for the two-dimensional irreducible representations of the double group at the zone centre in white tin

$\Gamma_6^+\ (\Gamma_5^+)$	$(x+iy)\,z\,\lvert-\rangle$ $(-x+iy)\,z\,\lvert+\rangle$
$\Gamma_6^-\ (\Gamma_5^-)$	$(-x+iy)\,\lvert-\rangle$ $(x+iy)\,\lvert+\rangle$
$\Gamma_7^+\ (\Gamma_5^+)$	$(-x+iy)\,z\,\lvert-\rangle$ $(x+iy)\,z\,\lvert+\rangle$
$\Gamma_7^-\ (\Gamma_5^-)$	$(x+iy)\,\lvert-\rangle$ $(-x+iy)\,\lvert+\rangle$

At the point Γ the only non-trivial representations of the double group are the two-dimensional ones $\Gamma_6^{\pm}$, $\Gamma_7^{\pm}$. The basis functions for these representations can be chosen in the form given in Table VI, where

$$u_{\varkappa}(\mathbf{r})\,|+\rangle + v_{\varkappa}(\mathbf{r})\,|-\rangle = \begin{bmatrix} u_{\varkappa}(\mathbf{r}) \\ v_{\varkappa}(\mathbf{r}) \end{bmatrix}. \tag{8}$$

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THEORETICAL PHYSICS

Is Planck's Constant a Constant in a Gravitational Field?

by

L. INFELD

Presented on June 19, 1961

Introduction

In spite of the simplicity of the derivation of the red-shift, I believe that its theory requires some deepening and critical investigation. To do so is the purpose of this paper. In this way we shall gain some insight into the question as to whether h is a constant in a gravitational field.

The usual theory involves the following assumptions: the atom is a mechanism emitting light with a certain frequency. This mechanism is characterized everywhere by the same ds^2 where ds is the time interval of the internal changes in the atom. We have

$$ds^2 = g_{00}\, dt^2,$$

since we assume only time variation and no change in the space coordinates of the radiating atom that is

$$dx^k = 0, \quad k = 1, 2, 3, \quad dx^0 = dt$$

(c = velocity of light = 1).

Now we identify

$$\frac{1}{dt^2} = \nu^2,$$

where ν denotes the frequency which we assume constant everywhere along the zero geodesic.

In the first approximation we compare $\nu_{(s)}$, the frequency sent from the surface of the sun, with $\nu_{(0)}$, the frequency sent from Minkowski space. However, both these frequencies are observed in the Minkowski space, that is at infinity, where observation is independent of the coordinate system. Now we have

$$ds^2 = g_{00(s)} \frac{1}{\nu^2_{(s)}}$$

which implies

$$\frac{\nu_{(s)}}{\nu_{(0)}} = \sqrt{g_{00(s)}}$$

giving the red-shift.

The red-shift in the classical theory

We would like to start our consideration with a few critical remarks on the above derivation. Firstly: how do we know that ν remains constant along the zero geodesic? Secondly: how do we know that ds^2 remains the same for an atomic process on the sun and in Minkowski space?

As to the first question, the answer was given by Laue [1]. We shall give it here in a somewhat changed and simplified form. Assume that the presence of an electromagnetic field does not change the gravitational field. Assume further that the radiation permits the application of geometrical optics. This means that the vector potential A_α ($\alpha = 0, 1, 2, 3,$) is

$$A_\alpha = a_\alpha e^{i\varphi} ,$$

where φ is a space-time function;

$$\left| \frac{\partial \varphi}{\partial x^\alpha} \right| = |\varphi_{|\alpha}| \quad \text{is very large.}$$

Then Maxwell's equations are reduced to the one eikonal equation

$$g^{\alpha\beta} \varphi_{|\alpha} \varphi_{|\beta} = 0 .$$

We call

$$\varphi_{|0} = \frac{\partial \varphi}{\partial x^0} = \nu$$

the frequency. We assume the space to be static. And now a few remarks about the definition of a static field and a static coordinate system. A static field is a field for which, in a special coordinate system, its metric takes the form

$$ds^2 = g_{00}\, dt^2 + g_{mn}\, dx^m\, dx^n; \quad m, n = 1, 2, 3,$$

where $g_{\alpha\beta}$ depends only on the space coordinates; that is

$$g_{\alpha\beta|0} = 0, \quad g_{0k} = 0; \quad \alpha, \beta = 0, ..., 3, \quad k = 1, 2, 3 .$$

A coordinate system such as that chosen above, representing a static field, is called a static coordinate system. We write

$$ds^2 = V^2\, dt^2 - d\sigma^2$$

which splits up the metric into time and space, $d\sigma^2$ being the metric form of the space. If we wish to pass from one static coordinate system to another, and if V^2 remains of the form

$$V^2 = 1 + \psi, \quad \psi \ll 1$$

the only time transformation is

$$t' = t + a, \quad a = \text{const}$$

and the space transformation must be independent of x^0:

$$x'^m = x^m (x^n).$$

Thus the change in time is well defined in such a coordinate system and V^2 behaves like a scalar. Its physical meaning follows from the fact that for a light ray we have

$$V^2 = \left(\frac{d\sigma}{dt}\right)^2 = (\text{velocity of light})^2.$$

Under these assumptions it can be shown that:

(a) a line perpendicular to the surface $\varphi =$ constant is a zero geodesic line.

(b) $\nu = \nu_0$ is constant along this geodesic line.

First let us prove (a). We call $x^\alpha(p)$ (where p is an arbitrary parameter) a line perpendicular to the surface $\varphi =$ constant, if it satisfies the condition

$$\varphi_{|\alpha} = \lambda \frac{dx_\alpha}{dp} = \lambda \frac{dx^\sigma}{dp} g_{\alpha\sigma},$$

where λ is an arbitrary function of x^α. From the condition $g^{\alpha\beta}\varphi_{|\alpha}\varphi_{|\beta} = 0$ we find by differentiating with respect to x^σ:

$$0 = g^{\alpha\beta}{}_{|\sigma}\,\varphi_{|\alpha}\,\varphi_{|\beta} + g^{\alpha\beta}\,\varphi_{|\alpha\sigma}\,\varphi_{|\beta} + g^{\alpha\beta}\,\varphi_{|\alpha}\,\varphi_{|\beta\sigma}$$

from which follows that

$$\frac{d\left(\lambda \dfrac{dx_\sigma}{dp}\right)}{dp} + \frac{1}{2}\lambda g^{\alpha\beta}{}_{|\sigma}\frac{dx_\alpha}{dp}\frac{dx_\beta}{dp} = 0\,.$$

This is the equation of the zero geodesic line which goes into the standard form if we replace dx_α by dx^α.

Now, we can easily show that (b) is true. This follows from the above equation of the geodesic line. Putting $\sigma = 0$ in a static coordinate system we have

$$\frac{d}{dp}\left(\lambda \frac{dx_0}{dp}\right) = 0$$

and

$$\lambda \frac{dx_0}{dp} = \varphi_{10} = \nu = \text{constant along the geodesic.}$$

As to the second of our questions, whether ds^2 can be regarded as a constant, we refer to the fairly exhaustive discussion in Eddington's well known book on Relativity.

The theory of the red-shift and quantum theory

Let us look at the red-shift from a different point of view — that of quantum mechanics.

As we know, the photon moves along a geodesic. If we choose $p = t$ and denote

$$\frac{dx^\sigma}{dt} \quad \dot{x}^\sigma\,,$$

its equation of motion is

$$\frac{d}{dt}(\mu\,\dot{x}^{\sigma}) + \frac{1}{2}\mu\, g^{\alpha\beta}{}_{|\sigma}\,\dot{x}_{\alpha}\,\dot{x}_{\beta} = 0.$$

According to the theory of photons we have

$$\mu\,\dot{x}^{\sigma} = h\nu\,\dot{x}^{\sigma},$$

here $\mu\,\dot{x}^{\sigma}$ is a four-vector. If the frequency ν is to agree with its definition, then ν must be a constant along the geodesic line. Because of our equation of the geodesic

$$\mu\dot{x}_0 = h\nu\,\dot{x}_0 = h\nu\,\dot{x}^0\,g_{00} = h\nu\,V^2 = \text{const}$$

since $\dot{x}^0 = 1$ and ν is a constant, we have

$$hV^2 = h^{(0)} = \text{const.}$$

This result follows quite rigorously from our definitions and it shows that if h is defined in the most obvious way it cannot be a constant, but it changes slightly in a gravitational field according to the formula

$$h = \frac{h^{(0)}}{V^2} = \frac{h^{(0)}}{1+\psi}.$$

The theory of the red-shift follows very simply from our reasoning. We assume, with de Broglie, that for a wave of matter the same relation

$$\mu\dot{x}^{\sigma} = h\nu\dot{x}^{\sigma}$$

is valid, where $\dot{x}^{\sigma}$ is the velocity of the wave packet and μ equals [2]

$$\mu^2\,g_{\alpha\beta}\;\dot{x}^{\alpha}\,\dot{x}^{\beta} = m^2 = \text{const.}$$

We assume that for an emitting mechanism $\dot{x}^k = 0$. For a wave packet emitted at the sun we have

$$\mu_{(s)} = \frac{m}{V} = h\,\nu_{(s)}.$$

Therefore

$$\nu_s = mV^2/Vh^{(0)} = \frac{m}{h^{(0)}}\,V = \nu_{(0)}\,V = \nu_{(0)}\sqrt{1+\psi},$$

which gives the theory of the red-shift in a simple way.

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EXPERIMENTAL PHYSICS

Angular Distributions of Fast Neutrons Elastically Scattered on Ca

by

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Presented by M. DANYSZ on June 3, 1961

The main purpose of this paper is to show that some results concerning elastic scattering may be obtained with neutrons having a continuous energy spectrum.

Angular distributions of neutrons elastically scattered on Ca were measured for neutron energies above 1 MeV. This limit of neutron energy was due to high background below this energy. The upper limit was determined by the falling off of the neutron intensity.

Experimental method

A collimated neutron beam from a WWR-S reactor running at 2 MW thermal power was scattered on a cylinder of natural Ca, 3 cm. high and 2 cm. in diameter.

Ten plateholders with Agfa K2 nuclear plates with emulsion 100 μ thick were disposed on the periphery of a circle drawn around the scatterer. The radius of the circle, i.e. the distance from the axis of the scatterer to the internal edges of the plates, was 15 cm., each plate subtending a plane angle of about 10°. The plates were inclined 10° from the horizontal plane and were placed at 30°, 60°, 90°, 120° and 150° from the direction of the incident neutron beam.

The exposition time was 50 hours.

Other plates were exposed in the same way but with the Ca scatterer removed in order to obtain information on the background.

After processing the plates by the usual method, proton tracks were counted in every plate on a standard area. Tracks with dip angle below 6° and deviation less than $\pm 60°$ (in the projection on the plane of the plate) from the radial direction were taken into account.

Results and discussion

The results are shown in Figs. 1 and 2. The errors indicated are statistical. These results may be compared with the predictions of the compound nucleus theory.

This theory implies that the signs of the matrix elements for the formation and decay of the intermediate states are uncorrelated. This random sign approximation gives symmetry of angular distribution about 90°.

Since about 1953 a number of experimental data have shown a marked discrepancy from expectations based on this theory. Paul and Clarke [1] have found that for 14 MeV neutrons the (n, p) cross sections are in many cases significantly larger than those predicted by the compound nucleus theory. The deviations were

most noticeable for medium and heavy nuclei. Austern, Butler and Mc Manus [2] proposed to take into account an additional mechanism involving mainly interaction with surface nucleons without the formation of a compound nucleus. This has made possible the explanation of the observed cross sections and led to the expectation that energy and angular distributions should also differ from the predictions of the compound nucleus theory.

Fig. 1

Fig. 2

Thus, symmetry of the angular distribution in respect to the plane perpendicular (in the C.M. system) to the incident beam is a characteristic feature of the compound nucleus theory, whereas the existence of an asymmetry may be interpreted as evidence of a contribution from direct interactions.

As may be seen from the results there appears a forward asymmetry. The ratio of neutrons scattered through 30° to those scattered through 150° is

$$\frac{\sigma_{30^\circ}}{\sigma_{150^\circ}} = 1.47 \pm 0.18 .$$

This ratio was computed for the whole neutron energy range 1-3 MeV and seems to indicate that below 3 MeV direct interactions contribute to the mechanism of scattering. Similar results were obtained recently by Lane, Langsdorf, Monahan and Elwyn [3] who used monoenergetic neutrons from the $^7Li\,(p, n)^7Be$ reaction and proportional counters as neutron detectors.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

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ТОМ IX Резюме статей ВЫПУСК 8

Г. МИЛИЦЕР-ГРУЖЕВСКАЯ, **ОСНОВНОЕ ОСОБОЕ РЕШЕНИЕ ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ УРАВНЕНИЙ** . . . . стр. 565—570

Настоящая работа посвящена точному анализу свойств основного особого решения (см. определение (2. 1) в работе [1]) параболической системы уравнений с частными производными и коэффициентами Гельдера.

А. ВИВЕГЕР, **НЕКОТОРЫЕ ПРИЛОЖЕНИЯ СМЕШАННОЙ ТОПОЛОГИИ К ТЕОРИИ ДВУНОРМНЫХ ПРОСТРАНСТВ** . . . . стр. 571—574

В работе приводятся примеры приложений введенной ранее автором „смешанной топологии" к исследованию двунормных пространств. В пкт. 1 даются некоторые замечания о продолжении γ-линейных функционалов. В пкт. 2 доказано, что теоремы Шмульяна и Эберлейна справедливы для γ-польных двунормных пространств.

М. АЛЬТМАН, **МЕТОД НАИСКОРЕЙШЕГО ОРТОГОНАЛЬНОГО СПУСКА**
стр. 575—580

В работе приводится итерационный метод решения линейных уравнений в пространстве Гильберта. Этот метод по существу является методом типа наискорейшего спуска. Характерной чертой этого метода является то, что для исправления приближенного решенеия, получаемые поправки ортогональны к свободному вектору уравнения. Вообще говоря, метод сходится быстрее, чем метод наискорейшего спуска.

М. АЛЬТМАН, **ОБОБЩЕНИЕ МЕТОДА ЛАГЕРРА НА ФУНКЦИОНАЛЬНЫЕ УРАВНЕНИЯ** . . . . стр. 581—586

В работе дается обобщение известного метода Лагерра нахождения корней полиномов. Вместо корней полиномов ищутся нулевые элементы нелинейных функционалов в пространстве Банаха. Для решения функциональных уравнений этого типа получается итерационный метод третьего порядка. Для исследования полученного метода применяется мажорантный принцип.

Г. ПЛЕБАНСКИЙ, **АЛГЕБРАИЧЕСКИЕ СВОЙСТВА АНТИСИММЕТРИЧЕСКИХ ТЕНЗОРОВ** . . . . стр. 587—591

В настоящей работе приводятся в компактном виде алгебраические свойства антисимметрических тензоров в четырехмерных пространствах (электромагнитное поле).

Г. ПЛЕБАНСКИЙ, **О НЕКОТОРЫХ ПРИМЕНЕНИЯХ АЛГЕБРАИЧЕСКИХ СВОЙСТВ АНТИСИММЕТРИЧЕСКИХ ТЕНЗОРОВ** . . . . стр. 593—597

На основании предыдущей работы, в которой исследовались алгебраические свойства антисимметрических тензоров, рассматриваются некоторые простые применения, между прочим, применения проекционных операторов для случая движения электрона в постоянном поле.

В. ЗЕНТЭК, **МАГНИТНЫЕ СТРУКТУРЫ В ГЕКСАГОНАЛЬНОЙ РЕШЕТКЕ** стр. 599—604

На основании развитого автором квантового метода, проводится анализ магнитной структуры ферромагнитных анизотропных монокристаллов, обладающих гексагональной симметрией. Показано, что в данном кристалле возможны структура Блоха или-же структура Ландауа-Лифшица в зависимости от размеров образца вдоль гексагональной оси. Найден точный критерий, позволяющий определить — который из этих типов имеет место в каждом конкретном случае.

Кроме того, получены улучшенные модели доменной структуры.

З. ГАЛЯСЕВИЧ, **О ВОЛНОВОЙ ФУНКЦИИ НЕНОРМАЛЬНОГО СОСТОЯНИЯ СИСТЕМЫ ФЕРМИ** . . . . стр. 605—607

Получены формулы для волновой функции системы Ферми-частиц в крайнем случае анизотропных, аксиально-симметрических проводников (спаривание частиц с параллельными спинами).

Показано, что основное состояние является S-состоянием. Одно- и двухчастичные состояния зависят от направления волнового вектора и являются P-состояниями.

М. МИОНСЕК и М. СУФФЧИНСКИЙ, **ПРОСТРАНСТВЕННАЯ ГРУППА БЕЛОГО ОЛОВА. IV. ФУНКЦИИ БАЗИСА ДЛЯ НЕПРИВОДИМЫХ ПРЕДСТАВЛЕНИЙ В ТОЧКАХ СИММЕТРИИ** . . . . стр. 609—615

В работе приводятся симметризованные плоские волны в точках симмметрии зоны Бриллюэна белого олова. В центре зоны сдаются двухмерные функции базиса двойной группы с аппроксимацией тесной связи.

Л. ИНФЕЛЬД, **ЯВЛЯЕТСЯ ЛИ ПОСТОЯННАЯ ПЛАНКА ПОСТОЯННОЙ В ГРАВИТАЦИОННОМ ПОЛЕ?** . . . . стр. 617—620

В работе рассматривается вопрос о физическом воздействии гравитации на квантовую постоянную Планка. Предполагается новое определение постоянной h: постоянная h — это соотношение между массой фотона и его частотой

$v - 1/dt$, где t — мировое время в системе статических координат. Доказывается зависимость так определенной постоянной h от пространственной точки. Хорошо известная формула для Ред-шифта получается как частный результат упомянутых рассуждений.

К. МАЛУШИНСКАЯ, Л. НАТАНСОН, И. ТУРКЕВИЧ и П. ЖУПРАНСКИЙ, **УГЛОВЫЕ РАСПРЕДЕЛЕНИЯ СКОРЫХ НЕЙТРОНОВ УПРУГО РАССЕЯННЫХ НА Ca**
стр. 621—623

Произведены измерения угловых распределений скорых нейтронов, упруго рассеянных на Ca под углами 30°, 60°, 90°, 120°, и 150° (в лабораторной системе кординат). Нейтроны детектировались при помощи фотопластинок Агфа К 2 толщиной 100 микронов.

Результаты измерений показывают, что для энергий нейтронов ниже 3 *Мэв* наряду с процессом образования составного ядра имеет место прямое воздействие.

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